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Full lenght resaerch paper

Visualization of geometrical problems and its influence on the strategy adopted by the learner and the facilitator in secondary school in Lebanon

Naim El Rouadi and Noha Al Husni

Associate Professor- University Of Balamand Faculty of Arts and Social Sciences Department of Education
Instructor – University Of Balamand Faculty of Arts and Social Sciences Department of Education

Corresponding Author Email: nhusnie@gmail.com

ABSTRACT

This study aims at creating an interactive instructional strategy that varies in application according to the type of Geometrical problems solved in the Secondary Grade levels in Lebanon. When solving real life problems, this involves visualization and implementation of the correct mathematical language in addition to analysis to reach informal deduction on Pierre Van Hiele Scale .On Alain Kuzniak Scale, solving real life problems includes the natural geometry or intuition/experience and the natural axiomatic geometry or hypothetical deductive laws. The basis of learning theories behind such strategy are two: The Zone of Proximal Development in Socio-Constructivism as per Lev Vygotsky and the Scaffolding in the Cognitive Theory as per Jerome Bruner without forgetting Rene Descartes' contribution in suggesting that the main key to solve a problem is by breaking it into smaller ones (Problem reduction or decomposition).Our strategy drives the learner to narrate his solution as a story line. In this paper, a real life problem started for Grade 10level with facilitators' intentions to solve it as a Geometry practice or application; it turned out through narrative problem solving to admit three other different solutions implementing Analytical Geometry, Trigonometry, and Elementary Algebra despite the fact that each of the solutions can be discussed following the didactic contract in different Grade levels to conform with the sequence requirements of K-12 curriculum in Lebanon. In the reflection phase of narrative analysis, the contributors synthesized the importance of geometry to visualize the situation through a geometric drawing ahead of looking for different solutions to the problem. Facilitators also emphasized the importance of narrative problem solving on facilitators' level and on learners' level.

Keywords: Visualization, Narrative problem solving, Development of Geometrical thinking, Scaffolding, Zone of Proximal Development.

INTRODUCTION

In an attempt to reinforce the development of geometric thinking, a group of facilitators started the discussion on the geometric solution of aG10-level-problem posed by one of them. The facilitators' work followed the narrative problem solving strategy which can be a story line whose scenario and sequence of events follow the dialogues and discussions between the facilitators or exchanging and analyzing each other's reasoning in approaching the solution(s). The facilitator who posed the problem had in mind a geometric solution that requires simple skills acquired in Grade 9 as properties of similar triangles and Pythagoras Theorem. The event seems familiar and happens every time

Facilitators meet to discuss curriculum development and subject matter. Many underestimate the importance of Such discussions and attend because it is required by their schools. Others consider it as a playground for facilitators to show off selfishly and arrogantly or consider them as boring lectures imposing new ineffective methodologies.

Few appreciate their real values concerning development of facilitators, learners and consequently the curriculum as a whole. Are such meetings classified as a real waste of time or do they alleviate facilitators' instructional design and lesson planning strategies and consequently have a positive impact on the transfer of knowledge to learners?

From a positive point of view our hypothesis is in favor of supporting such meetings and discussions as long as all participants are lenient and interactive especially when the meetings are held in a narrative analysis ambient.

Subject Matter under Discussion

Back to the discussed problem by the facilitators, what follows is a presentation of the situation problem and the Geometric Solution as posed by one of the facilitators and adopted by the other contributors.

Situation

The two houses of Albert and Marcel are 2.6km apart. Albert's house (A) is at 700m distance from the rectilinear rails. Marcel's house (M) is 1km farther than Albert's house distance from the rails. The two houses of Albert and Marcel are equidistant from the train station S (SA = SM).

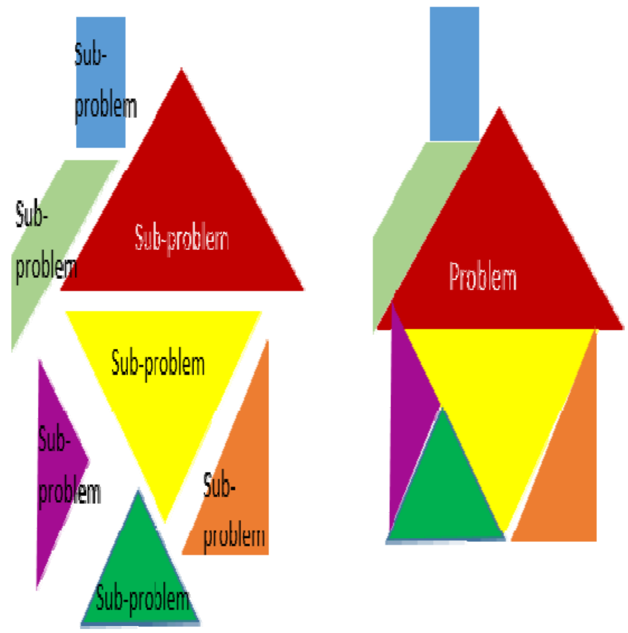
Problem

Knowing the location of the train rails and Albert's house, where is Marcel's house, where is the train station and what is the distance between the station S and either house?

Strategy Adopted

Rene Descartes' Contribution to our strategy

According to Rene Descartes this is a complex problem that needs to be decomposed into smaller and easier problems in order to be solved "Problem reduction or decomposition" (Grabiner, J., 1995):



Start with understanding the problem to convert it into given data and then represent it by a geometric figure(.a)The train rails are rectilinear parallel lines and the station S is just next to one side of the rails where the distance between the rails and the station is considered zero. In addition, the station should also be on perpendicular bisector of the segment [AM] in order to be equidistant from either house.

a) The 0.7 km distance of Albert's house from the rails is the shortest distance to the rails meaning the perpendicular distance; and similarly the 1.7 km distance of Marcel's house from the rails is also the shortest or perpendicular distance from the rails.

b) The rails are two parallel lines separated by a small distance; in this problem the two rails are represented by one line assuming the distance between the two rails (1.435m) negligible relative to the distances 700m, 1.7 km, 2.6 km…

c) The two houses A and M can be located on the same side of the rails

d) The two houses A and M can be located on opposite sides of the rails

In solving such a problem, all the previous ideas should be taken into consideration in order to account for all possible solutions.

Reasonable assumptions and considerations are necessary:

- The train station is a point S on the rails.

The two rails are considered as one straight line (Y).

- On the geometric drawing designate the Station, Albert's house, and Marcel's house by the points S, A and M respectively.
- Use geometric tools (ruler, set square and a compass) to construct an accurate figure.
- If figure results in symmetric solutions, one of them will be calculated and the other follows the same proof.

Therefore a strategy is to be adopted by the facilitator who will coach the learners to progress through solving the problem and this can be clarified by asking several questions to the learners in order to support them in their mission as follows:

The Geometric solution for Grade 10:

Question 1)

How can you locate geometrically by construction the position of pointM which is at 1.7 km distance from the rails and at the same time at 2.6 km from point A?
Narratively the answer to this question follows as such:

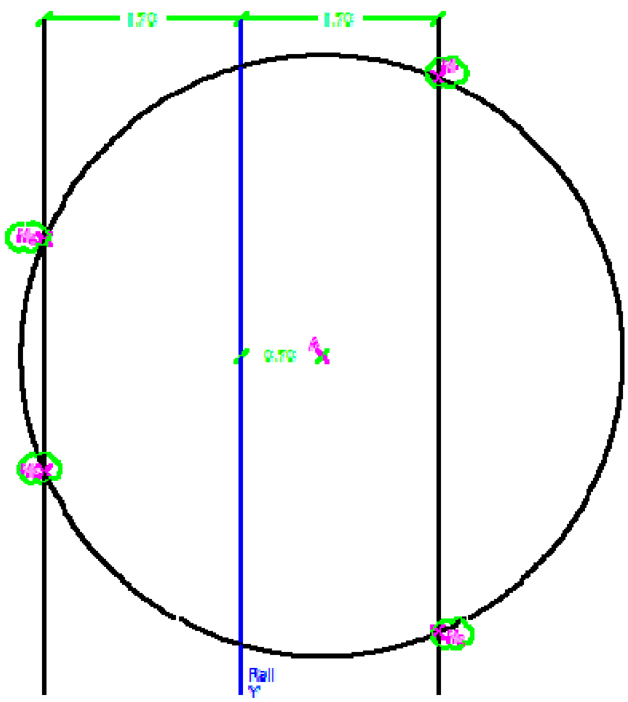


Figure 1

Since M is at 1.7 km distance from the rails, then M belongs to a line parallel to (Y)from either side at 1.7 km.
M also is 2.6 km away from point A, then M belongs to a circle (center A and radius 2.6 km).

Then M is the intersection point(s) between the line(s) and the circle.

As shown on the adjacent figure, there are four possible positions for M (indicated on the figure as M1, M2, M3, and M4).

Question 2)

Show that the 4 points of M formtwo symmetrical pairs.
AM1 = AM2 = R = 2.6 km, and
AM3 = AM4 = R = 2.6 km.

Then the triangles AM1M2 and AM3M4 are isosceles, the perpendicular bisector of [M1M2] and [M3M4] passes by A and it is the diameter of the circle (center A, radius R=2.6 km), keep in mind that [M1M2] and [M3M4] are parallel to the rails (Y).

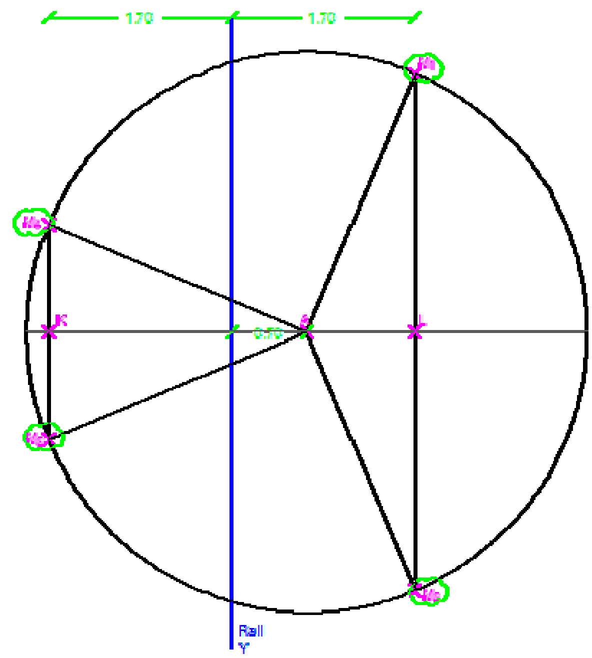


Figure 2

Therefore, the 4 points are symmetrical pairs: (M1, M2) and ($M_{3,}$ M_4).

Question 3) A is given, M (one case at a time) is located; determine the location of point S and calculate SA and SM.

Consider the cases of M_1 and M_2 where A and M is to the same side of the rails (Y); calculate the length of [S'A] and [S'M_2] knowing that S'A= S'M_2
Since M_1 and M_2 form the symmetric pair solutions; M_2 is considered in this case as shown on figure.

The intersection of the perpendicular bisector of [AM2] and the Rails (Y)is the train station point S'

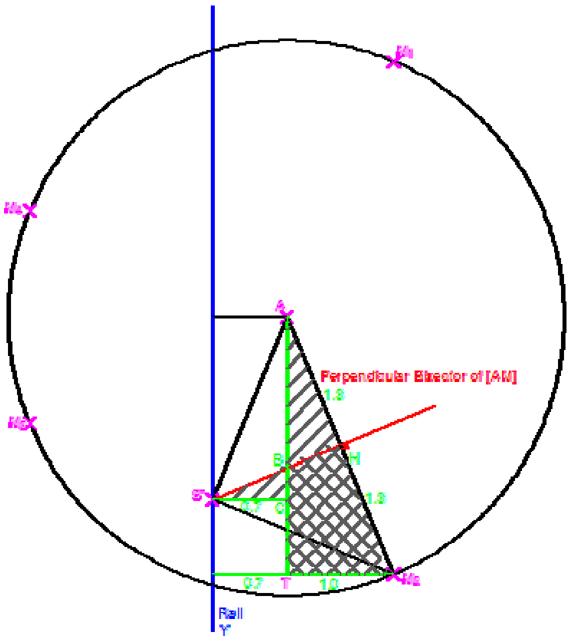


Figure 3a

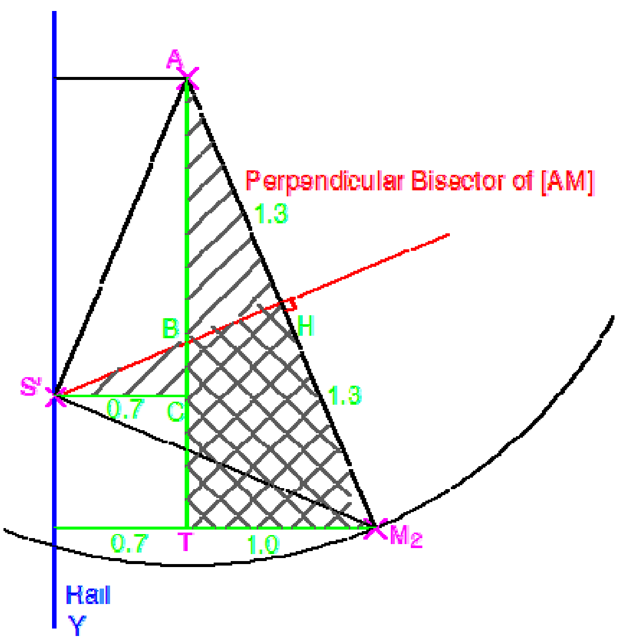


Figure 3b

.Triangles ATM_2 and AHB are similar triangles since the angles at A are the same and the angles at H and T are right.

Then, $\frac{AT}{AH} = \frac{AM_2}{AB} = \frac{TM_2}{HB}$ and $\frac{2.4}{1.3} = \frac{2.6}{AB} = \frac{1}{HB}$ and

AB = $\frac{1.3 \, x \, 2.6}{2.4} = \frac{1.3^2}{1.2}$ and HB = $\frac{1.3}{2.4}$

Triangles AHB and S'CB are similar triangles since the angles at B are vertically opposite and the angles at H and C are right.

Then,

$\frac{AH}{S'C} = \frac{AB}{S'B}$ substitute for numerical values $\frac{1.3}{0.7} = \frac{\frac{1.3^2}{1.2}}{S'B}$ and S'B = $\frac{1.3 \, x \, 0.7}{1.2}$

Therefore, S'H = S'B + BH = $\frac{1.3 \, x \, 0.7}{1.2} + \frac{1.3}{2.4} = \frac{1.3 \, x \, 0.7 \, x \, 2 + 1.3}{2.4} = \frac{1.3 \, x \, 2.4}{2.4}$ = 1.3 km

Then triangle HAS' is a right isosceles triangle since AH = S'H = 1.3 and consequently,

Applying Pythagoras Theorem: $(S'A)^2 = (AH)^2 + (S'H)^2 = (1.3)^2 + (1.3)^2 = 2 \, x \, (1.3)^2$

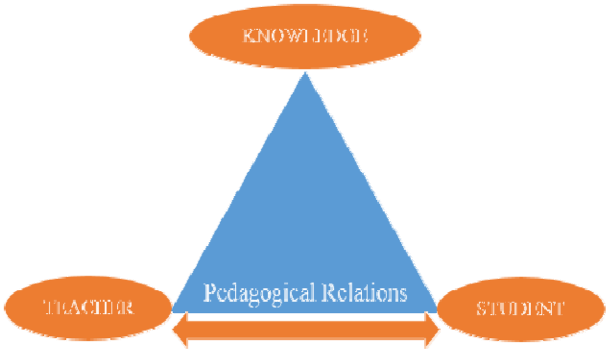
Then S'A = $1.3\sqrt{2}$ = 1.838 km.

Didactic Contract presence and influence on our strategy

Based on the didactic contract between facilitator and learner, other approaches to the solution were suggested by the learners and guided by the facilitators; Indiogine Henri-Paul, (2010, p.1) translates the definition of 'didactic situation' set initially by Guy Brousseau (1997) as: "It is the set of the reciprocal obligations and sanctions that each partner in the didactic situation imposes, of believers to impose, explicitly or implicitly, on others, and those that are imposed on him or her, or he or she believes that they are imposed on him or her," Indiogine, H. also translates from a document of Brousseau (2003, p. 2) a definition for didactic situation as "where anagent, the teacher for example, organizes an intervention that manifests its intention to modify the knowledge of another agent or causes it arise. The second agent, for example, is a student that is allowed to express him or herself in actions."

"The didactical contract is described by Brousseau (1998) as the set of the rules that determines what the students and the teacher "have the responsibility to carry on, and what each one is responsible in some way"(Brousseau, 1998, p. 61). Extending this definition, Sadovsky (2005) describes this contract as a keen game in which the teacher communicates "sometimes explicit and many other times implicitly, through words and also through gestures, attitudes and silences, aspects related to the functioning of the mathematical affair that is treated in the class (p. 37).

So that during this process, "meanings are negotiated, mutual expectative are transmitted, methods of performing are suggested or inferred, mathematical norms are communicated or interpreted (in an explicit or implicit way)" (p. 38)." Retrieved in Arias, F. & Araya, A. (2009, p. 1 &2).



DIDACTIC TRIANGLE

The presented definitions all focus on the pressures and tensions experienced by the learner and the facilitator in the classroom while solving a mathematical problem. As a result, the learners attempt to tie what has been learned, grasped, acquired to the problem under study. Since in grade 10 learners study Geometry and Trigonometry, the two related solutions were suggested by the Grade 10 learners and discussed until the final resolution is accomplished. Similarly in grade 11, learners implemented the Analytic Geometry and the Elementary Algebra to reach the same solution but through a different track. The solutions admitted are shown in different appendices at the end of this paper as indicated below:

- Refer to Appendix A to see the figures for the solution of point M_1, the symmetric case of M_2 at the end of the article.
- Refer to Appendix A for the solutions of the other pair (M_3, M_4) symmetrical cases.
- Refer to Appendix B for the other three solutions of the problem using Trigonometry, Analytic Geometry and Algebra
- Refer to Appendix C for the rejected

Pedagogical Interpretation of the adopted strategy to the presented solution:

The presented geometric solution seems complicated due to the number of steps required to prove the right isosceles triangle and then deduce the distance between the station S and either of the houses A or M although, as mentioned previously, the implemented theory is Pythagoras in addition to the properties of similar triangles which are two skills acquired in Grade 9 level according to K-12 curriculum in Lebanon.

Through the narrative problem solving strategy, it was discovered that the other facilitators and later the learners in the classroom were able to solve the same problem using different approaches that cover Trigonometry, Analytic Geometry, and Elementary Algebra. Consequently, the problem is resolved in four different methods. The difference between them is the student level according to the K-12 curriculum. One of the other solutions which implements Trigonometry can be covered in Grade 10 level too while the other solutions lifted the learners' level to Grade 11 in order to use Analytic Geometry and Elementary Algebra.

The impact of Narrative Problem Solving, talk moves or think aloud on the presented strategy:

In "Chapter2: The Tools to Classroom Talk", 2003-©Math Solutions Publications, the authors define the 'talk moves' as a strategy in class used to support mathematical thinking; talk moves help in organizing the students' conversations inside the classroom in an evenhanded, respectful environment motivating all students to participate in an interactive mood. They explained five different productive talk moves:

1) Re-voicing which permits for thinking space; in re-voicing the facilitator repeats using different terminology what the learner said giving time for all to understand what was said and to think if it is correct or not.

2) Restating the reasoning of other learners; as one of the learners shares his reasoning with the class, the facilitator asks other learners to restate his reasoning to be sure that others are taking his reasoning seriously and Consequently motivates the learners to contribute in a comprehensible manner.

3) Asking students to use their reasoning on other learner's claim or reasoning; in other words, driving the learners to agree or disagree with their classmates and at the same time justify why.

4) Encouraging students for further participation; always ask the learners if they like to add anything to what have been said. The learners will be motivated to add an idea or an interpretation or a justification expressing their thoughts.

5) Using wait time; as facilitator asks leading questions or tries to drive the learners to proceed, s/he is supposed to wait for the students to arrange their thoughts and at the same time this gives the non-super-fast thinkers in class a chance to think and participate.

The listed productive talk moves are very crucial in our strategy, the facilitator starts with the presentation of the real life problem and by posing the leading questions, the facilitator using re-voicing, restating, reasoning, encouraging participation, and the waiting time for thinking can reach two important starting levels of solving the problem which are the comprehension of the given data and what is required in addition to setting the necessary assumptions

before drawing the geometrical figure. (Refer to the adopted strategy and the geometric solution above).
In the same chapter the authors highlight three talk formats in class:

The whole-class discussion where the facilitator is in charge but to coach and not to provide answers, The small group discussion which gives a chance for 3 to 6 students to discuss a question together and come out with one answer from the group; again the facilitator plays the role of observer to assure that the group discussion is on the right track, and The partner talk in which every two adjacent learners turn and discuss the question together; this gives chance for most of the learners to think out loud, analyze others reasoning and come out with a convincing answer.

The three types of talk formats are encouraged in the classroom to support the learners in discovering the solutions of the symmetrical case, the other pairs (Appendix A), the Trigonometric, Analytic Geometry, and the Elementary Algebra approaches to solve the problem (Appendix B). These talk formats were also useful in eliminating the rejected solutions of having the two houses A and M on the same line perpendicular to the Rails (Y) whether from the same side of the Rails or from opposite sides of the Rails. (Appendix C)

As explained the narrative problem solving has one major objective: involving all the students in the running activity turning the classroom into an interactive stage for all learners to collaborate in an equitable and respectful environment that encourages all learners to exchange their thoughts.

Narrative Problem Solving can hold other meanings as narrating a story including the problem situation and the ambiguity to trigger the reader's curiosity to proceed or the narrative problem solving does not provide answers but leads the thinking to deduce a solution step by step and so on. (Baily, J., 2007& Hakkarainen, P. n.d)

Our adopted strategy integrates all the aspects of narrative problem solving described above whether applied to facilitators in discussing all the solutions or to learners in the classroom in order to discover all the other solutions starting from symmetrical cases to similar cases in geometry then shifting to other mathematical fields as Trigonometry and Analytic Geometry to reach finally the Elementary Algebra solution.

To continue with the interpretation of the adopted strategy: Concerning the development of Geometrical thinking, analyzing the presented problem indicates clearly that all the solutions start from visualizing the problem using the Mathematical tools and language. This starts with the learners' ability to translate the problem situation into a set of points, lines, circles, triangles… and to draw an accurate geometric figure using the necessary geometric tools as ruler and compass. Without visualizing such a figure, none of the four approaches is possible.

Moreover, such talk moves in the classroom will lead the learner to master the mathematical language necessary to communicate with others in order to express his suggested solution and to interpret others reasoning. This language unifies the common ground for the learners to interact.

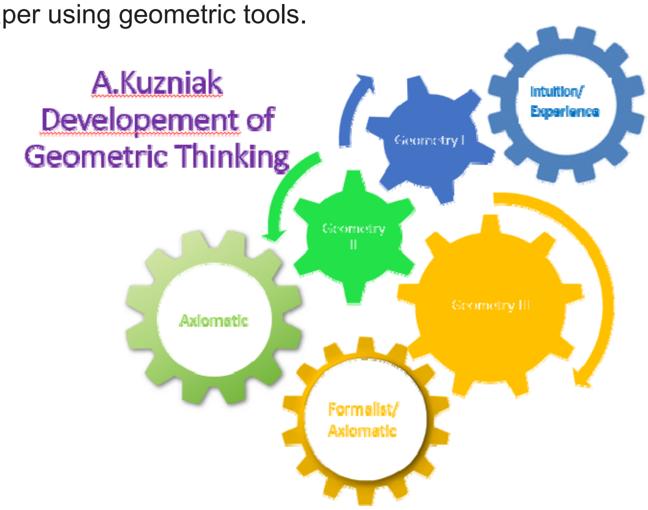
In addition to the informal deductive level reached by the learners as they shift from one step to another with correct mathematical justification. It is not just inspired by their intuition; learners can back-up their reasoning with properties, theorems and rules that they can implement at the right time in the correct step. These ideas of geometrical thinking development are supported by two researchers: Alain Kuzniak and Pierre Van Hiele. What follows summarizes both theories:

Alain Kuzniak and Pierre M. Van Hiele divided the development of Geometrical Thinking into three and five levels respectively; each researcher tried to highlight on the difficulties of acquiring essential geometric skills through the K-12 curriculum. The problem is worldwide and many articles/papers were written and projects/case studies were conducted in attempt to find solutions and consequently help the learners at different phases of their learning journey.

Alain Kuzniak levels of developing geometrical thinking

According to A. Kuzniak, learning geometry occurs through back and forth movement between three levels of Geometry labeled as Geometry I, Geometry II, and Geometry III.

In brief, Geometry I/ Natural Geometry is the first level in which the learner's reasoning depends mainly on his intuition and experience; his reasoning is based on natural real facts and manipulation using tools and instruments during experimentation. For example, if you ask the learner to construct a 4cm x 8cm x 10cm triangle, s/he may first use sticks of specific lengths to construct the triangle or may be able to draw the triangle on a paper using geometric tools.



Geometry II/ Natural Axiomatic Geometry is a higher level where the learner can justify the existence of a geometric form through axioms and hypothetical deductive laws which are the closest possible to his intuition and reality around him/her. For example, at this level a combination of side-lengths may seem odd or may not exist as 4cm x 4cm x 10cm which requires the implementation of properties between the sides of a triangle to exist.

Geometry III/ Formalist Axiomatic Geometry is the highest level according to Kuzniak; it involves a disconnection between reality and axioms thus abstraction. This level manipulates cases that do not necessarily exist in real life; for example Charles' Theorem which applies to triangles through vector relations and do not necessarily have an application from real life. (Houdement, C & Kuzniak, A., 2003)

According to the K-12 curriculum, only Geometry I and II levels are required to be achieved and gained by the learner in the presented problem while Geometry III level is covered at University level. Kuzniak permits transitions between the levels as much as the learner needs to accomplish a certain task. The table below summarizes the skills of each level as viewed by Kuzniak:

Table 1: A.Kuzniak Scale *Ref: TG7_Houdement_cerme3*

	Geometry I (Natural Geometry)	Geometry II (Natural Axiomatic Geometry)	Geometry III (Formalist Axiomatic Geometry)
Intuition	Sensible, linked to the perception, enriched by the experiment	Linked to the figures	Internal to mathematics
Experience	Linked to the measurable space	Linked to schemas of the reality	Logical
Deduction	Near of the Real and linked to experiment	Demonstration based upon axioms	Demonstration based on a complete system of axioms.
Kind of spaces	Intuitive and physical space	Physical and geometrical space	Abstract Euclidean Space
Status of the drawing	Object of study and of validation	Support of reasoning and "figural concept"	Schema of a theoretical object, heuristic tool
Privileged aspect	Self-Evidence and construction	Properties et demonstration	Demonstration and links between the objects. Structure.

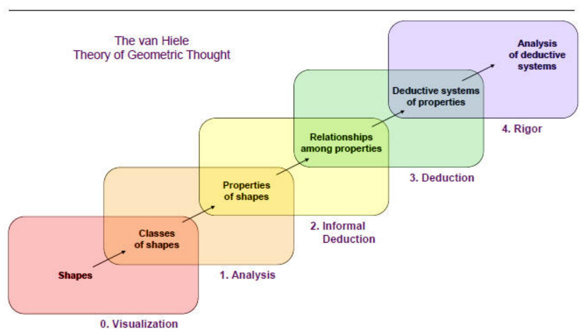
In relation to presented problem, the learners intuition and experience help them to symbolize the real-life problem into Geometric drawing; they can imagine the rails as the straight line (Y), each of the houses as one point A or M, they are able to draw the necessary lines to visualize the distances SA and SM whose lengths to be determined. As the learners distinguish right and isosceles triangles on their figure, measurable and physical space is encountered. The aforementioned in correspondence with the above table are covered under the natural geometry level. After reaching this level of thinking, the move between Natural Geometry and Axiomatic Natural Geometry is triggered. According to the table above, learners start using and manipulating the geometric forms obtained with their properties (similar triangles and Pythagoras theorem….) to demonstrate their approach and support their reasoning in order to finally through this back and forth movements between Geometry I and Geometry II levels reach a resolution for the problem.

Pierre M. Van Hiele levels

The Van Hiele's model for development of geometric growth embeds five sequential levels: Visualization, Analysis, Informal Deduction, Deduction, and Rigor (Crowley, Mary L., 1987; Marchis, I., 2012; Clements, D. 2003; & Van Hiele, P. 1999):

Level I – Visualization/ Non-verbal thinking level: this level covers recognizing shapes and forms holistically with some vocabulary to label the shapes without involving the learner in the details, properties, and rules. At this level, the learner can reproduce a shape/figure. Van Hiele in this level came across the "spatial thinking" or ability to manipulate mentally two- and three-dimensional figures. Level II – Analysis/ The Descriptive level – at this level the learner starts analyzing the geometric figures dismantling them into parts and elements as angles, sides… The learner judges a figure by its properties rather than "it looks like one". The learner uses a specific language to describe the shapes but he cannot yet put an order to the properties of one shape.

Level III - Informal Deduction/ Abstract-Relational level – this is the highest level required in K-12 curriculum. The learner can logically order the properties of shapes and use acquired information about the shapes properties to come up with definitions for such shapes.



Van Hiele Theory of Geometric thinking
Reference: http://proactiveplay.com/the-van-hieles-model-of-geometric-thinking/

Level IV – Deduction/ Axiomatic Proof: The learner at this level is able to use his knowledge and establish geometric theory in an axiomatic system. The learner is also capable of understanding a theory and its converse.

Level V – Rigor/Abstract Level: The learner can study in different axiomatic systems such as the Non-Euclidian Geometry. The learner can compare between different systems and lifts Geometry to its abstract level.

Again with Van Hiele, the learners in the level of Visualization, they are able to imagine the real life problem as a schema and consequently draw the geometric figure which represents their visas to search for applicable solutions. When the Analysis level is reached, the learners recognize that the geometric forms encountered in this problem are different types of triangles (right triangles and isosceles triangles) which in turn lead them to start thinking of the different properties and theorems applicable to triangles to pick out of which those of similar triangles and pythagoras theorem. Reaching the third level, learners implement the chosen properties to approach a final solution for the problem.

The two researchers did not reach paradoxes in their discoveries but on the contrary, their findings converge to insure each other's approaches. A comparison done by Alain Kuzniak himself was done and presented in the table below:

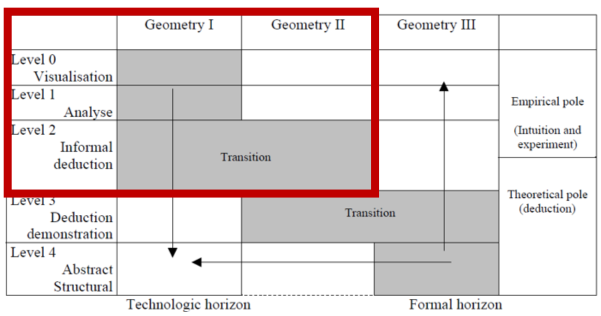


Table 2: The intersection between Kuzniak and Van Hiele Scales applicable to problem
Ref: TG7_Houdement_cerme3

The table shows:
1 The applicable zone to school level and consequently to the presented problem and the adopted strategy to solve it is within the red frame.
2 The first three levels of Van Hiele's intersects with Kuzniak Geometry I and II phases. Visualization and Analysis are completely included in Geometry I while Informal Deduction is a transition between Geometry I and Geometry II.
3 Geometry I and Geometry II are mainly the Empirical Pole that implement Intuition and Experiments and allow a small room for Theoretical Pole or the deduction.

In addition to what have preceded, this strategy requires a well-designed lesson plan to be prepared by the facilitator following the Zone of proximal development - ZPD- as per Lev Vygotsky. Vygotsky, in the socio-constructivism learning theory or socio-cultural theory, argues that a learner solving a problem with the guide or aid of the More Knowledgeable Other (MKO) can progress through a much wider margin to reach the region where he can perform similar and more complicated tasks on his own. The More Knowledgeable Other can be a peer learner, a parent, the computer with all its possible utilities or the facilitator as in our case. Most probably, if the learner tries to do the task on his own while still in his ZPD, he would fail and consequently blocks his geometry learning. Therefore, the facilitator is to be well prepared for all the blocking obstacles to learners and ready to offer a hint or a question or else to guide the learners to the correct path that leads finally to cognitively solve the problem. As the learners proceed from one step to a higher step, they must be confident of their cognition of what is solved and what is still required to be solved.

ZPD defined by Lev Vygotsky and how it relates to this strategy

Christmas, D., Kudzai, Ch., and Josiah, M. (2013) in their research about Vygotsky's Zone of Proximal development included an interesting literature review about the subject tracing the interpretations of several other researchers like Roosevelt- 2008,Murray, & Arroyo – 2002, Rogoff – 1990, O'Neil - 2011, Ross – 2001, Well – 2000, and many others.Christmas, D. et Al indicated that the ZPD according to Lev Vygotsky does not stand alone but includes two other important concepts as "Mediation" and "Scaffolding." In their opinion, ZPD which can improve mathematical achievements, is the zone where active learning takes place with the help of the MKO.

Zone of Proximal Development ZPD*Reference-* *http://1.bp.blogspot.com/-q2H0hjVtnGw/UKEwbAVPCKI/AAAAAAAAAC8/tUF6TrrAugA/s1600/ZPD.jpg*

In their exact words on page 2: "According to Vygotsky, the zone of proximal development is the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or collaboration of more capable peers". Specifically in learning mathematics, "it determines the level of work to be taught which is desirable to the child."

Since the hardest phase on novice in solving a real life problem is the interpretation of the given data and acknowledging the required to be determined in order to find a solution or more for the problem, it is obvious that the instant learners read the story of the problem they are stuck with how to represent it by a geometric figure in order to move from real case to schematic representation to proceed afterwards. Just at this instant learners enter their ZPD's regions which vary from one learner to another; consequently, the facilitator's coaching is highly recommended at this step. By asking leading questions, the facilitator drives the students to represent the rails by one straight line (Y) and the houses of Albert and Marcel by two points A and M in addition to considering the station as one point S on the rails (Y) and so on; in addition to, accepting the necessary approximations and assumptions as presented in the Geometric solution above. As the learners construct the correct figure and are convinced that it is a good representation for the problem, they are motivated, encouraged and have the self confidence to proceed towards the solution(s).

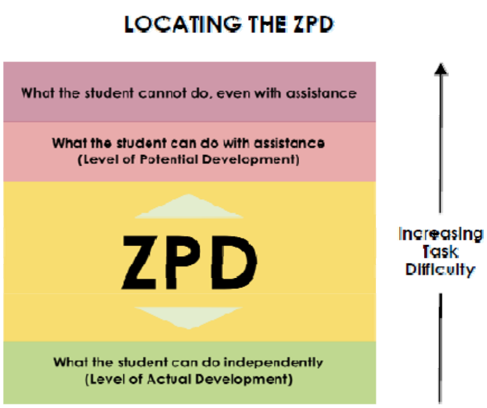
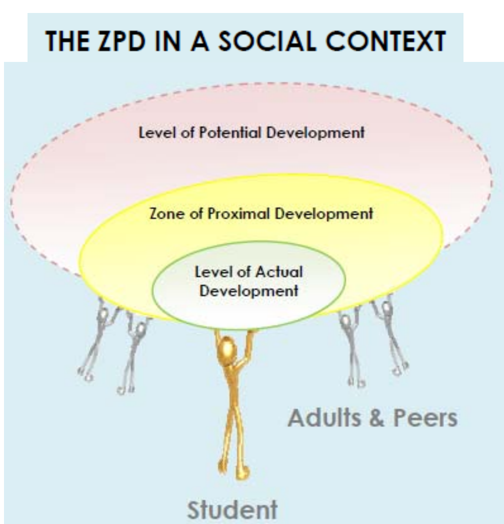


Figure and quote are adapted from "Teaching in the Zone" by Angela Lui – 2012, P. 3



The figure below gives a definition for Level of potential development, ZPD, and level of actual development:

Level of Potential Development:
Refers to what a student is not able to do independently but able to do under adult guidance or in collaboration with more competent peers. Teaching should not happen at this level as it would not result in effective learning.

Zone of Proximal Development (ZPD):
Refers to the area between the level of actual development and the level of potential development. Also known as the *instructional level*, this is where instruction should be focused to drive the greatest learning gains for each student.

Level of Actual Development:
Also referred to as the *independent level*, it encompasses skills a student has already mastered and can perform independently. Instruction focused here would not provide a sufficient challenge for students.

Figures are adapted from "Teaching in the Zone" by Angela Lui – 2012, P. 4

As to mediation, it is the nucleus of the socio-cultural theory of Vygotsky. It can be defined as "the part played by other significant people in the learners' lives, people who enhance their learning by selecting and shaping the learning experiences presented to them" (Christmas, D. et Al, 2013. P3) indicating that mediators are tools to the learner and one of the crucial tools is the language which is an important pillar to socio cultural theory. The language as a tool helps the novice to translate his thoughts into sentences using the appropriate vocabulary of Mathematics. This converges to the previously explained levels of development as per Van Hiele and Kuzniak.

Here also the facilitator plays an important role in directing the learners to use the correct mathematical vocabulary acknowledging their semantics and properties; i.e. without knowing what a straight line definition is, the learner cannot recognize its analogy to the train rails and without knowing the exact meaning and shape of triangle (right or isosceles), the learners cannot draw the necessary connecting lines to determine the isosceles triangle formed by the station S, Albert's house A, and Marcel's house M; and so on. This involves also the think time discussed in the narrative problem solving above.

Scaffolding or targeted assistance, the third concept after ZPD and Mediation, although as a term was suggested by Wood, Bruner, and Ross (1976 and 2001), "Educators and researchers have used the concept of scaffolding as a metaphor to describe and explain the role of adults or more knowledgeable peers in guiding children's learning and development." Retrieved in (Christmas, D. et Al, 2013. P4). Morrissey and Brown (2009) stated, "The aim of scaffolding is the ultimate transfer of responsibility for the task to the child as adult support decreases and child capability increases" (p.107).Another quote from Christmas, D. et Al, (2013): “Educationally, scaffolding is an instructional structure whereby the teacher models the desired learning strategy or task, then gradually shifts responsibility to the students.” (p.4)

The inclusion of Scaffolding in our strategy is obvious when the facilitator guide the geometric solution and then inspire the learners by the different questions and hints about the M_2-symmetric solution to case M_1 and increase the level to ask the learners for the solution of point M_3 and gradually its symmetry M_4 follows. Later, the facilitator opens a narrative discussion for implementing other mathematical fields of Trigonometry, Analytic Geometry and Algebra if none of the learners (the fast thinker in the class) suggest such solutions first. Finally, the facilitator can leave the learners on their own to solve the same problem with a small change for example consider M as fixed and the learners have to locate all the possible positions of A. All of the above suggestions are scaffolds for the learners to pass from a mastered level of learning to a higher one while crossing their ZPD margins and consequently, this shows how ZPD, Mediators, and scaffolding influence the development of the learners’ geometrical thinking thus transferring the responsibilities gradually from facilitator to learners.

Moreover, Lui, A. (2012), in her “Teaching in the Zone” - white paper focusing on the ZPD of Lev Vygotsky, wrote: “Vygotsky’s pioneering research focused on the role of social interactions in human development. In his investigation of cognition, he examined the complex relationships between language and thought as well as learning and play. What he found was that children’s social interactions with significant individuals in their lives (parents, peers, teachers, and other adults) profoundly shape their interpretations of the world and higher order thought processes” (p.4).“In order for teaching and learning to be effective, instruction should focus on skills and knowledge that are attainable for students (not too easy, not too difficult, but *just right*).” (Lui, A., 2012, p.5).

Although it is a challenge for the expert to determine and define each novice’s ZPD which differs from one learner to another, it is very helpful in supporting the development of mathematical thinking of learners. It is obvious after this review that the skills and strategies of the facilitators while learners are in their ZPD region empowers them to develop their mathematical thinking in a motivating interactive environment. Synthesizing the above impressive and accurate quotations lead to the conclusion: It is the facilitator’s Monopoly that counts in the lesson planning phase; s/he in charge, is the responsible in the classroom who can guide the novice and support them to be lifted to expert level.

This adopted strategy, is also in favor of what Jerome Bruner labeled as “Scaffolding “and the famous Cognitive Theory of Learning. Scaffolds are targeted hints and aids given to learners according to their needs at different phases of learning. Bruner argues that such coaching synergizes the learner’s efforts to reach the cognition expected from a specific task. Such cognition is a life-long lesson that counts in their thinking and cannot be lost as it doesn’t implement memory alone but analysis and inferences; in other words hypothetical informal deduction which again relates to Van Hiele and Kuzniak works.

Jerome Brunertheories implemented in this strategy

In fact, many ideas of Jerome Bruner’s contribution in Education and Learning will be discussed and how they relate to the adopted strategy. To start with the three ways for the novice to understand also called the modes of representations ((Bin Surif, J., n.d., Simply Psychology, Bruner. 2012, Research for teachers, 2006& Lutz,S. et Al. 2004)

1) The Enactive: The infant until 18 months old recognizes by physical actions and manipulations.

2) The Iconic: The child between 1.5 and 6 years old is able to form mental representations or images for the ideas that can be triggered for retrieval by hearing, smelling, touching, or gestures. The child already has her/his perceptions through the Iconic phase but as a visual memory.

3) The Symbolic: From 6 years old and through adulthood, understanding becomes possible through words or symbols consequently language (words, numbers, music…) that is verbal memory. This level applies to the adopted strategy where visualization and language are two important phases in order to construct a figure for the problem situation and to be able to communicate with facilitator and other learners to express clearly their thinking through the correct mathematical vocabulary and language.

The second idea in Bruner’s achievements is the transfer of knowledge which has a wide room in the cognitive theory. Under the spiral curriculum Bruner ensured the acquiring of knowledge by the learner at any age level as long as the skill is adapted to the development of the learner. Bruner also emphasized the necessity of revising what has been learnt every now and then to insure deeper understanding of the acquired skill or knowledge. (McKoy, J. 2011) In our strategy, the transfer happens through the talk moves of the narrative problem solving where repetition of the same idea by different learners ensures the understanding of all learners in an interactive environment. Moreover, when the learners are motivated to solve symmetrical cases, they are revising and engraving their acquired knowledge and consequently insuring transfer.

The third idea covers the inductive and discovery learning which both encourage active learning in addition to the construction and understanding of the concepts.

Inductive learning starts with a specific experiment and then with the coaching of the facilitator, the concept is consolidated with additional ideas as differences and similarities until a generalization is reached. Finally, the acquired concept is to be applied in a new context to insure the understanding or learning (McKoy, J. 2011).In our strategy, the geometrical solution is considered as the first experience to the learners and through guidance, learners acquire the knowledge but through further practices on similar cases (symmetric, cases of other solutions) the learner constructs self-confidence about mastered knowledge, and by fixing M and asking the learners to locate A, the learners are handling the problem in a new context. As explained, this strategy is covering all these multileveled aspects of developing the learners' mathematical thinking.

The discovery learning is harder and requires from the facilitator to prepare the environment and all facilities for the learner to discover the concept; the facilitator provides guidance and help whenever necessary. Through discovery learning, the noviceis supposed to "learn how to learn", to acquire the skills of problem solving through inquiry and apply the concepts in new situations in order to develop the learner's intellectual potentials. Discovery learning helps the learner to recall the knowledge acquired whenever needed. Discovery learning provides the learner with different procedures and encourages her/him to develop attitudes and practice strategies for problem solving in addition to increasing the learner's abilities to analyze, synthesize, evaluate and work with a team. Bruner advices to implement discovery learning through guided discovery by asking leading questions during the process of learning. In this approach the facilitator is supposed to be well prepared for the session to be conducted in order to save time and avoid learner's frustration and the discovery of the wrong concept.This leaves the facilitator with a big challenge to account for all the mentioned factors in order to succeed in his mission.Bruner also highlighted the influence of the social exchanges between the child and the adult.(Bin Surif, J., undated, McKoy, J., 2011, &Research for Teachers, 2006)

The last idea is the Scaffolding as Wood, Bruner and Ross defined it "a temporary support structure around that child's attempts to understand new ideas and complete new tasks. The purpose of the support is to allow the child to achieve higher levels of development by simplifying the task or idea, motivating and encouraging the child, highlighting important task elements or errors, and giving models that can be imitated."(McKoy, J., 2011. P8 and P9).

As presented above, the adopted strategy prepares the facilitator for coaching the class by discussing all the possible solutions in collaboration with other facilitators. S/He can predict most of the obstacles and misconceptions of the learners and can prepare the leading questions in the correct sequence in order to save time. When the learners are motivated to participate in the narrative problem solving or the "think aloud", they will discover the correct concept and ignore the wrong one with confident justification. The learners when searching for all the other solutions of Geometry, Analytic Geometry, Trigonometry and Elementary Algebra while the facilitator is always ready to aid through the leading questions, are increasing their intellectual potentials and driving up the higher levels of the spiral curriculum building on previous knowledge, revising concepts for symmetric solutions and discovering new solutions.

CONCLUSION

on teacher education courses the related concepts of Vygotsky's 'zone of proximal development' (ZPD) and 'scaffolding' (Wood, Bruner, Ross, 1976) can provide a useful framework which will equip teachers with the necessary strategies and skills to build up and gradually extend children's interactive and discourse skills in appropriate ways at different ages and stages of learning." (Read, C.2004. p.1).This quote is Read's hypothesis on teaching English language. Our strategy supports Read's hypothesis but when applied to a completely different field – mathematics.

The presented strategy which integrates all the discussed theories aiming to develop mathematical thinking in Secondary level demonstrates many advantages that can summarized by the following points:

The learner cannot jump over the visualization step as all the suggested solutions start from visualizing them on the accurate figure drawn even for the Elementary Algebra approach.

Facilitators and learners are convinced by practice that there is no one unique way to reach the solution of a situation problem as the presented problem admits four solutions in different mathematical domains Geometry, Trigonometry, Analytic Geometry and Elementary Algebra.

The presented mix of implementing the narrative problem solving with the levels of Developing learners' Geometrical Thinking of P. Van Hiele and A. Kuzniak in addition to the fabulous well planned coaching involving the ZPD, Mediations, and Scaffolding in the Socio-cultural theory of L. Vygotsky and the Cognitive theory of J. Bruner ensures a boost for learners in a motivating interactive mode of learning.

In conclusion, the adopted strategy explicitly demonstrates its influence on facilitators' level as they plan for their lessons and consequently on the learners' level providing them with just the right aid at the right time for the right age taking into consideration their acquisition of the visualization phase as the very first step in approaching any of the possible solutions.

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APPENDIX A:

Following the didactic contract, the different solution(s) prove that the knowledge is grasped by the learners through Practice, Metacognition, ZPD, and Scaffolding after exposing the learners to the Geometric solution as presented and discussed by the facilitator. This is considered as a proof since the mental representation is different as the drawing for the symmetrical case of M_2 is different and despite this change, the learners can still solve it. The figures below summarize the M_1- symmetrical case of Geometric solution for Point M_2.

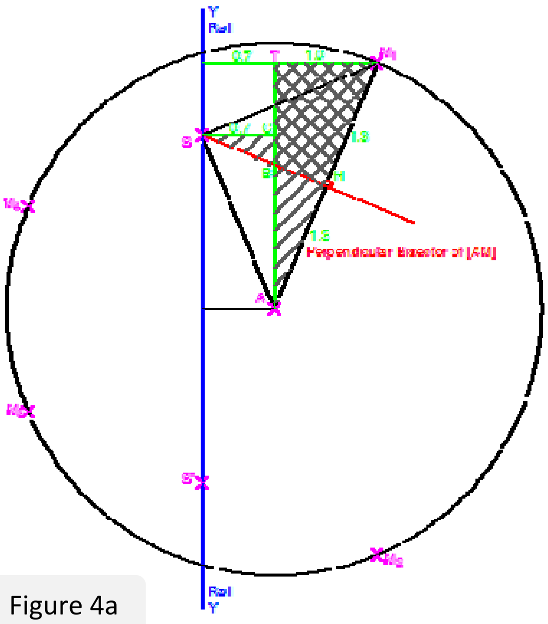


Figure 4a

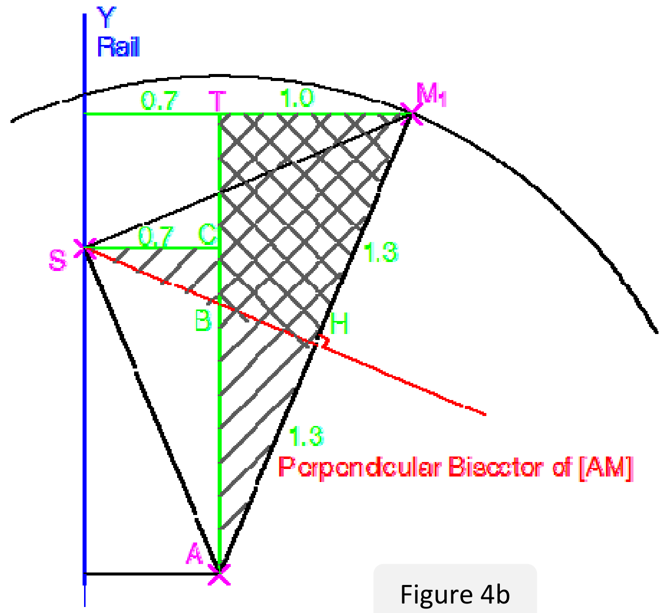


Figure 4b

Solving geometrically for the cases of M_3 and M_4:

Here also the learner justifies the higher level of thinking achieved by solving the cases of the other pair of points (M_3, M_4) – solutions based on the didactic contract. The learners can reproduce the geometrical figure for the other pair of points M and can do the necessary connections to obtain the triangles (right and isosceles), and then apply the properties of similar triangles and Pythagoras theorem to reach the expected solution. All those accomplishments consolidates the effectiveness of our strategy by investing in the integration of ZPD, Scaffolding and the Narrative Problem Solving criteria. The admitted approach is presented below:

Question: Continue the geometric solution in Grade 10 to cover all the possibilities.

Consider the cases of M_3 and M_4 where A and M are to opposite sides of the rails Y; calculate the length of [S'A] and [S'M_4] knowing that S'A= S'M_4

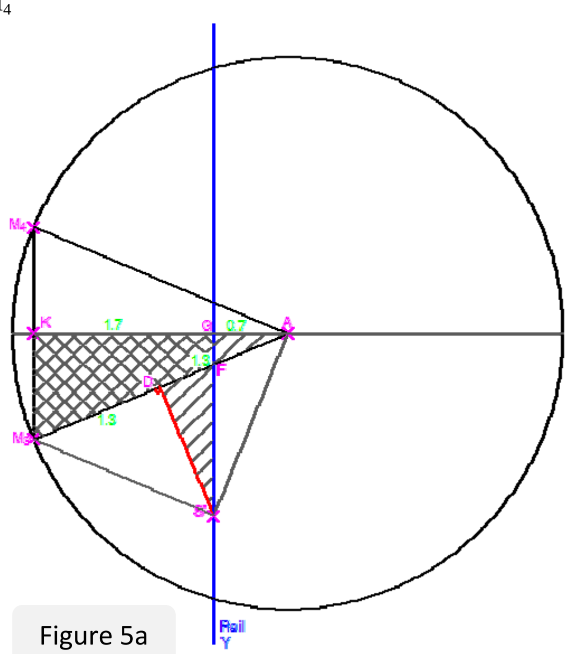


Figure 5a

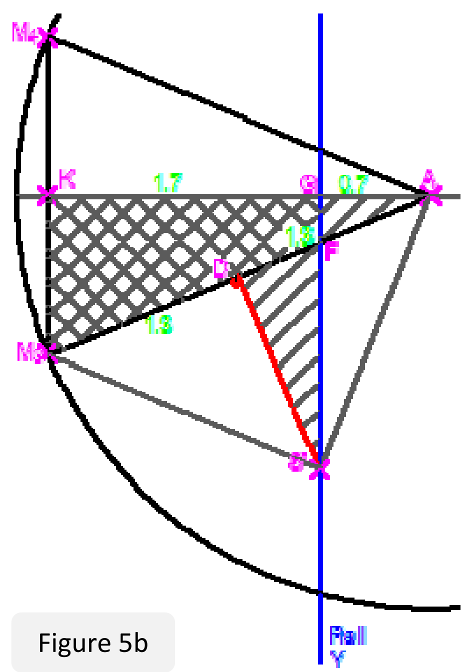


Figure 5b

Consider the right triangle AKM_3
$AK = AG + GA = 1.7 + 0.7 = 2.4$ km
$AM_3 = R = 2.6$ km
Apply Pythagoras theorem $(KM_3)^2 = (AM_3)^2 - (AK)^2$
And $(KM_3)^2 = (2.6)^2 - (2.4)^2 = 6.76 - 5.76 = 1$
$KM_3 = 1.0$ km (since it is a distance then the negative answer is ignored)
Triangles AGF and AKM_3 are similar since they have the same angle at A and equal right angles at G and K.
Then: $\frac{AG}{AK} = \frac{AF}{AM_3}$ and $\frac{0.7}{2.4} = \frac{AF}{2.6}$ and $AF = \frac{2.6 \times 0.7}{2.4} = \frac{1.3 \times 0.7}{1.2}$
$AM_3 = R = 2.6$ km
$M_3D = DA = 1.3$ km since S'D is the perpendicular bisector of $[AM_3]$
$DF = DA - AF = 1.3 - \frac{1.3 \times 0.7}{1.2} = \frac{1.3 \times 0.5}{1.2}$
Triangles AKM_3 and S'DF are similar triangles since both are similar to triangle AGF
(S'DF similar to AGF since angles at F are vertically opposite equal angles and the angles at D and G are both right.)
$\frac{AK}{S'D} = \frac{KM_3}{DF}$ and $\frac{2.4}{S'D} = \frac{1}{\frac{1.3 \times 0.5}{1.2}}$ and

$S'D = \frac{1.3 \times 0.5 \times 2.4}{1.2} = 1.3$ km
Therefore, triangle S'DA is right isosceles and $S'A = 1.3\sqrt{2} = 1.838$ km

As for the symmetrical case for point M_4, the following figure shows the geometric forms reached by the learner to solve it in a similar way to case of M_3:

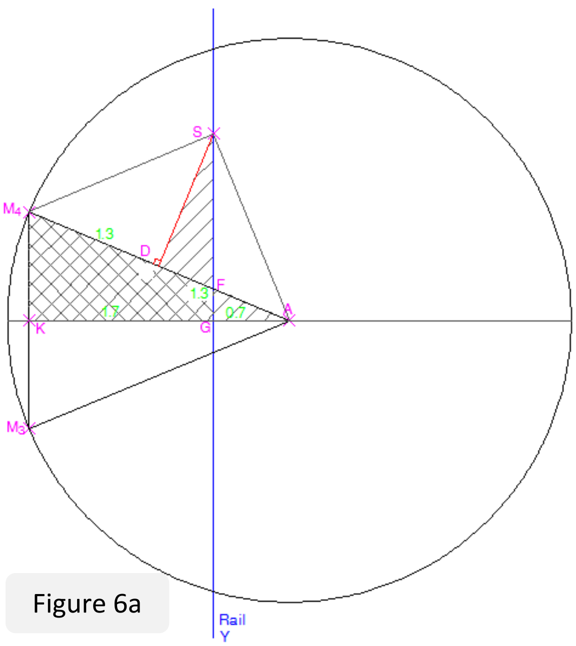


Figure 6a

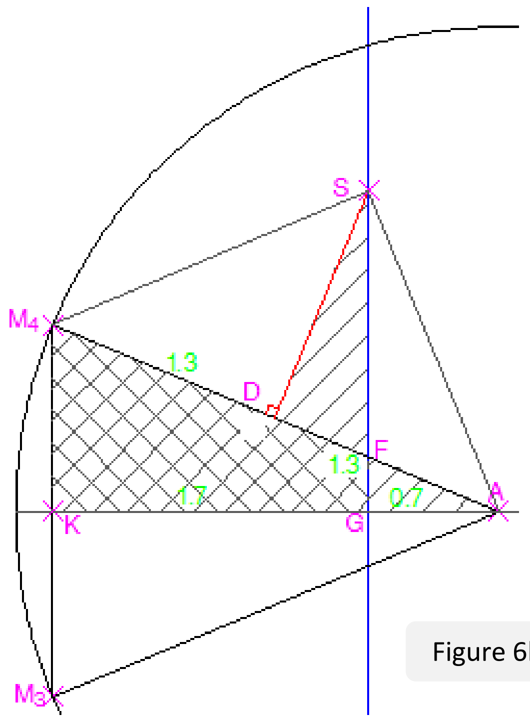


Figure 6b

APPENDIX B

A second solution is feasible that implements trigonometry to Grade 10 level according to K-12 curriculum:

As the didactic contract between learners and facilitator indicates, learners in grade 10 are studying trigonometry according to K-12 curriculum. As a consequence of our effective strategy, learners show development through the integration of ZPD (which lifts learning in the same domain) and Scaffolding (which allows the learner to expand into other mathematical domains as trigonometry) and they are able to use the same figures and manipulate on them to reach the trigonometric solution. What proceeds is the trigonometric approach as suggested by learners: Join M_1 and M_2 by a straight line, and (D) the parallel to Rail (Y) through point A. This parallel (D) intersects [S'H] in point B.

In triangle AM_2L, the angle $\widehat{M_2}$ is equal to the angle $\hat{A}$ in triangle HAB, (alternating angles).

In right triangle AM_2L:

$(LM_2)^2 = (AM_2)^2 - (AL)^2$

$(LM_2)^2 = (2.6)^2 - (1)^2$

And $LM_2 = 2.4$ km

$\tan \widehat{M_2} = \frac{AL}{LM_2} = \frac{1}{2.4}$

In the right triangle HAB,

$\tan \hat{A} = \frac{BH}{AH} = \frac{BH}{1.3}$

Then we deduce that: $\frac{1}{2.4} = \frac{BH}{1.3}$, and $BH = \frac{1.3}{2.4}$

The straight line (S'C) is a perpendicular to line (D) at point C.

The straight line (S'H) is perpendicular to (AM_2), and (SC) is perpendicular to line (D). Then the angle $\widehat{BSC}$ is equal to $\widehat{BAH}$ and equal to the angle $\widehat{AM2\,L}$.

In the right triangle CS'B: Cos $\widehat{S'} = \frac{S'C}{S'B} = \frac{0.7}{SB}$.

In the right triangle AM_2L: Cos $\widehat{M_2} = \frac{LM_2}{AM_2} = \frac{2.4}{2.6} = \frac{1.2}{1.3}$.

We deduce that: $\frac{0.7}{S'B} = \frac{1.2}{1.3}$, then $S'B = \frac{0.7 \: x \: 1.3}{1.2}$

S'H = S'B + BH

$S'H = \frac{0.7x1.3}{1.2} + \frac{1.3}{2.4} = \frac{(2x0.7x1.3\,)}{2.4} + \frac{1.3}{2.4} = \frac{(1.4+1)x\;1.3}{2.4}$

Therefore, $S'H = \frac{2.4 \; x \; 1.3}{2.4} = 1.3$ km

We conclude that the triangle AS'H is right isosceles at S'.

In right triangle AS'H,

$AS'^2 = S'H^2 + HA^2$

$AS'^2 = (1.3)^2 + (1.3)^2$,

$AS'^2 = 3.38$,

AS'= 1.838 km

And AS' = $S'M_2$ = 1.838 km

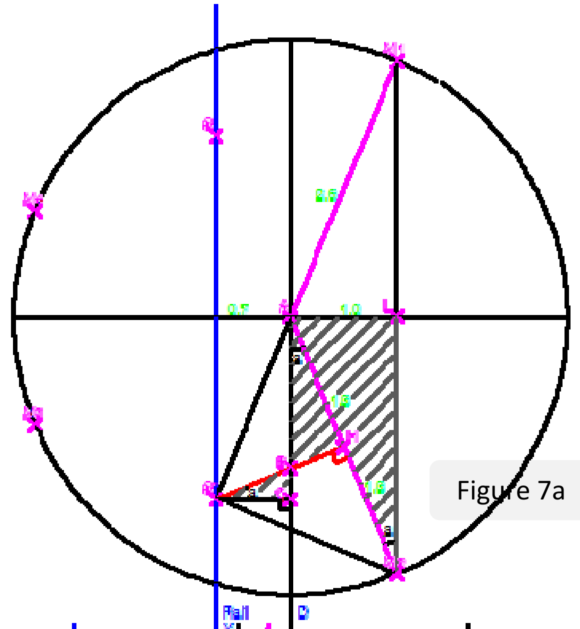


Figure 7a

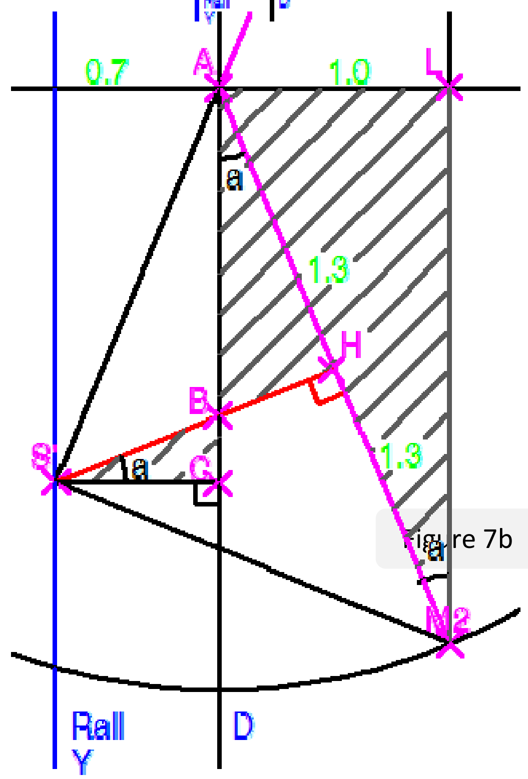


Figure 7b

Because Grade 11 learners are introduced to Analytic Geometry and Elementary Algebra, they in turn discovered their new approaches implementing the newly acquired knowledge in different mathematical domains. One more time this shows explicitly the influence of our strategy on the learning process which in accordance with the didactic contract and by interweaving ZPD characteristics and Scaffolding practices through a Narrative Problem Solving approach lead the learners to acquire the knowledge in high levels of thinking and develop their mathematical thinking by shifting from Classical Geometry to Modern Analytic Geometry and Elementary Algebra. Two other approaches follow:

A third solution is also feasible but involves analytic geometry and consequently Grade 11 - a higher level according to K-12 curriculum:

Consider the orthonormal system of axes $(A, \vec{\imath}, \vec{\jmath})$ where $\vec{\imath}$ is along $\overrightarrow{AL}$ and $\vec{\jmath}$ is along line (D).

Then the circle center A and radius R = 2.6 km has the equation: $x^2 + y^2 = (2.6)^2$

And the line joining points M_1 and M_2 has the equation x = 1 and the line joining M_3 and M_4 has the equation x = -1.
To determine the coordinates of the four points M_1, M_2, M_3, and M_4, solve for the intersection between the circle and the two lines x = 1, x = -1 and $x^2 + y^2 = (2.6)^2$

a) $x = 1$ and $x^2 + y^2 = (2.6)^2$
   $(1)^2 + y^2 = (2.6)^2$
   $1 + y^2 = 6.76$
   $y_2 = 6.76 - 1 = 5.76$
   $y = +/- 2.4$
   Therefore the coordinates of M_1 (1, 2.4) and M_2 (1, -2.4)

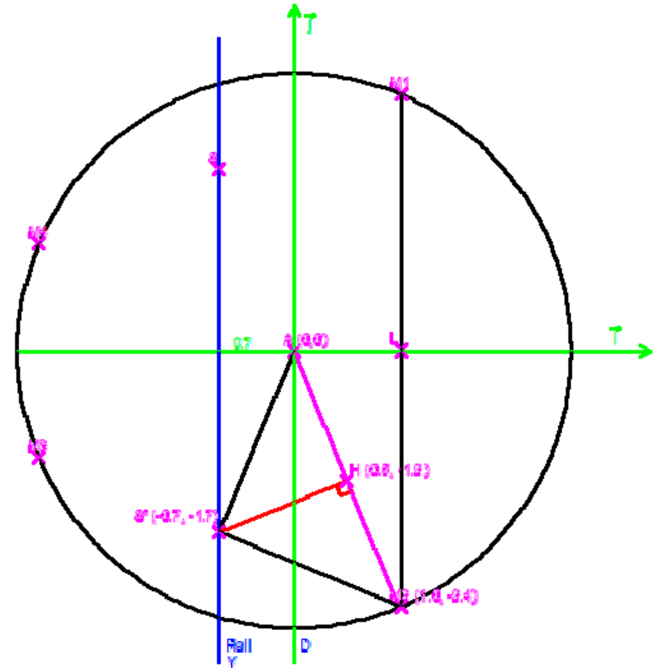


Figure 8a

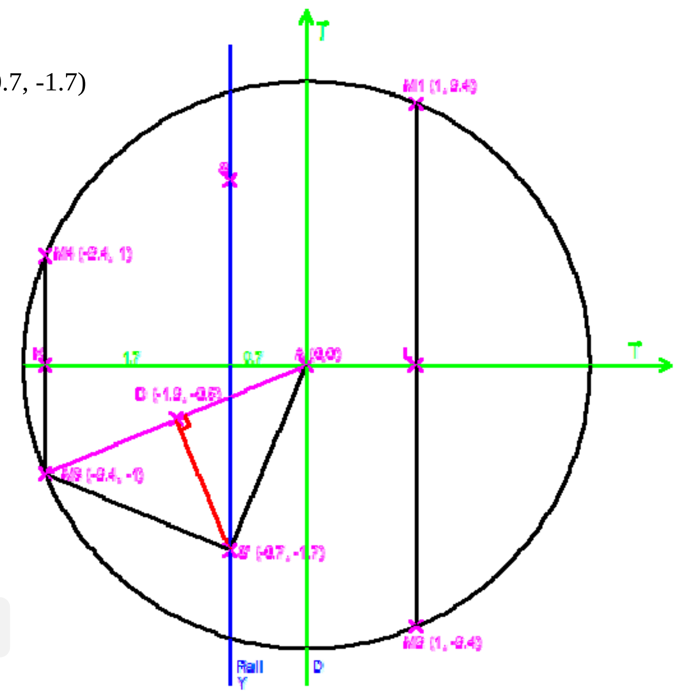
For A (0, 0) and M_2 (1.0, -2.4)
Their midpoint H has the coordinates:
$x_H = \frac{1}{2}(x_A + x_{M_2}) = \frac{[illegible]}{2} = 0.5$
And $y_H = \frac{1}{2}(y_A + y_{M_2}) = \frac{0-2.4}{2} = -1.2$

Then H (0.5, -1.2)
Let S' (-0.7, y) be the train station; calculate y.
S'H is the perpendicular bisector of AM_2, then the scalar product of $\overrightarrow{AM_2}$ and $\overrightarrow{S'H}$ is zero.

In the orthonormal system (A, $\vec{i}$, $\vec{j}$),
$\overrightarrow{AM_2} = (x_{M_2} - x_A).\vec{i} + (y_{M_2} - y_A).\vec{j}$
$\overrightarrow{AM_2} = (1.0-0).\vec{i} + (-2.4-0).\vec{j} = 1.0\vec{i} + -2.4\vec{j}$
And $\overrightarrow{S'H} = (x_H - x_{S'}).\vec{i} + (y_H - y_{S'}).\vec{j}$
$\overrightarrow{S'H} = (0.5+0.7).\vec{i} + (-1.2-y).\vec{j} = 1.2\vec{i} + (-1.2-y)\vec{j}$

$\overrightarrow{AH}.\overrightarrow{S'H} = 1.2 + 2.88 + 2.4y = 0$
$2.4y = -4.08$ and consequently, $y = \frac{-4.08}{2.4} = -1.7$; Therefore, S (-0.7, -1.7)
Then $(S'A)^2 = \sqrt{(0.7-0)^2 + (1.7-0)^2}$

$(S'A)^2 = 1.838$ km



b) $x = -1$ and $x^2 + y^2 = (2.6)^2$
   $(-1)^2 + y^2 = (2.6)^2$ and $1 + y^2 = 6.76$
   Then, $y^2 = 6.76 - 1 = 5.76$
   And consequently, y = 2.4

Therefore the coordinates of point M_3 are(-1, -2.4) and M_4 (-1, 2.4).
And the rest follows the same procedure to calculate S'A.

Figure 8b

The forth solution implements Elementary Algebra:

The solution starts from the same figure showing the lines to which point(s) A belongs. These lines are parallel to the rails Y at +0.7 km and -0.7 km. The figure also shows a second pair of lines parallel to the rails Y at +1.7 km and -1.7 km to which point(s) M belongs. Choosing a location for point A (any point on the parallel lines at +/- 0.7km, M belongs to the circle center A and radius 2.6 km.
M satisfies the equation of the circle in system (A, $\vec{i}$, $\vec{j}$) then M satisfies $x^2 + y^2 = (2.6)^2$
But the abscissa of M is x = -2.4 then $(-2.4)^2 + y^2 = (2.6)^2$ and $y^2 = (2.6)^2 - (-2.4)^2$
$y^2 = 6.76 - 5.76 = 1$. Therefore, y = +/- 1 km
Draw MH the perpendicular from M to Y; MH is1.7 km.

Draw AP the perpendicular from A to Y; AP is 0.7 km.

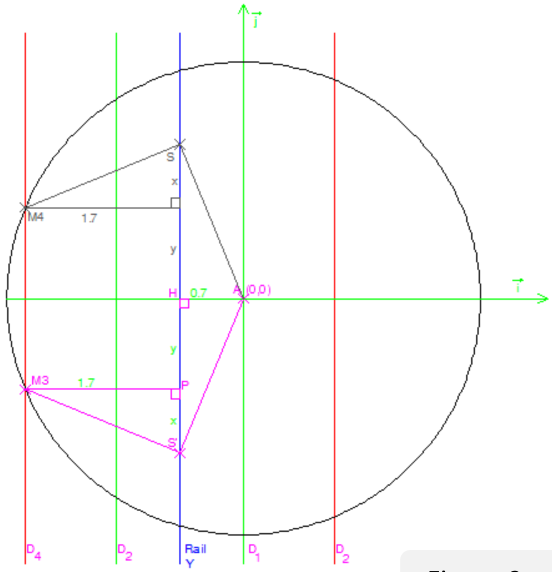


Figure 9a

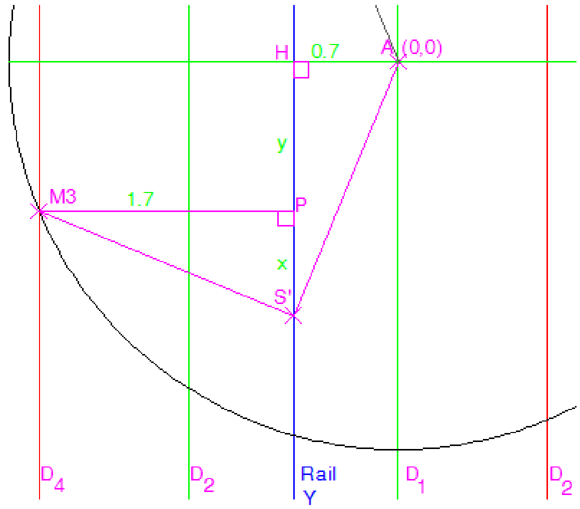


Figure 9b

The ordinate of point M_2 is y = +1 while the ordinate of point M_3 is y = -1, two symmetric solutions.
Consider Solution of point M_3: The distance HP is therefore 1 km and S'H is (x+1) km
Apply Pythagoras theorem:
In triangle S'M3P: $(S'M_3)^2 = (1.7)^2 + x^2$
In triangle S'AH: $(S'A)^2 = (0.7)^2 + (x+1)^2$
But $(S'M_3)^2 = (S'A)^2$
Then $(1.7)^2 + x^2 = (0.7)^2 + (x+1)^2$
A quadratic equation to be solved
2.89 + x2 = 0.49 + x2 + 2x + 1
1.4 = 2x and x = 0.7 km,
Consequently, $S'M_3 = S'A = \sqrt{2.89 + 0.49} = 1.838$ km

APPENDIX C

The Rejected cases:

The learners through their narrative problem solving came across two cases that they rejected with logical reasoning. The two cases are illustrated in the adjacent figures; they cover the primary thoughts that tangle the learners' minds when the houses of Albert and Marcel belong to the same perpendicular line to the rails whether on the same side of the rails (refer to figure a) or on opposite sides of the rails (refer to figure b). In both cases the distance between the two houses is not 2.6 km and it can be easily determined to be 1km in the case of figure (a) and 2.4 km in the case of figure (b). In addition, in both cases S cannot belong to the rails (Y) as it belongs to the perpendicular bisector of AM which is a line (in red) parallel to the rails (Y).

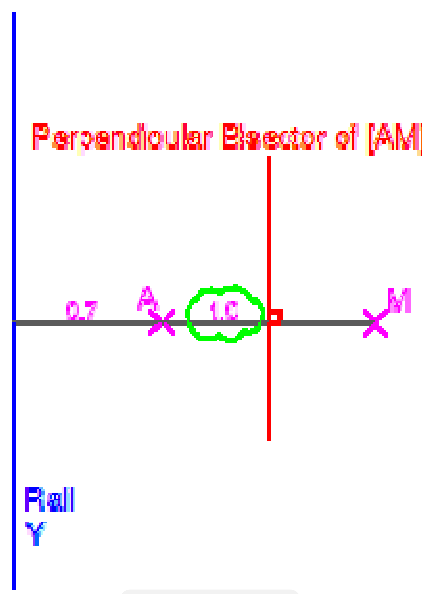


Figure 10 a

Such analysis done by the learners, increases their level of thinking through ZPD again it is reached first by visualizing the figures, then by proceeding to the reasoning phase and searching for justifications and properties to reject them.
Rejecting a hypothesis shows that learning took place as it involves reasoning through previously acquired knowledge of different theorems. This reflects a high level of thinking as it embeds not only the pedagogical relations between the learners and the facilitator but also all the characteristics of ZPD, Scaffolding, and narrative problem solving where the interactivity helps the learners to be confident decision makers about what to accept and what to reject.

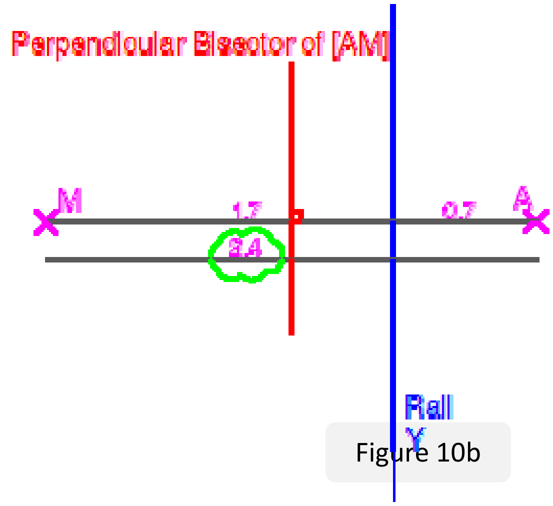


Figure 10b

Similar problem in a new context:

The same problem can be given with one difference: Marcel's house is known so point M is given and Albert's house is to be located by the learners. One more time, the learners are obliged to start with the visualization of the case, drawing the necessary connecting lines, obtaining the geometric figures of triangles, and proceed as in the previous cases until they resolve the problem. This figure also can hold the start of four different approaches in the mathematical field: Geometry, Trigonometry, Geometric Analysis, and Elementary Algebra.

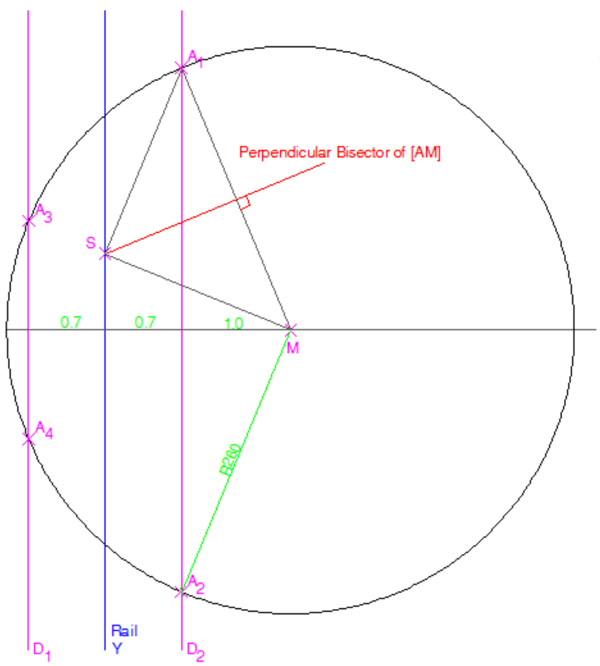


Figure 11