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*Full Length Research Paper*

# Time dependent model for chemically reactive pollutants emitted from various line sources into a stable atmospheric boundary layer

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**An analytical model for dispersion and diffusion of chemically reactive primary pollutants emitted from an elevated line sources into a stable atmospheric boundary layer with generalized wind velocity of quadratic function of vertical height z. The model obtained from an analytical solution of the atmospheric diffusion equation with the quadratic diffusion coefficient (exchange coefficient) and three different types' sources viz. continuous, an instantaneous and step-function type sources. The pollutants considered to be of chemically reactive primary pollutants emitted from the above said sources. The results are validated with that of the case where there is particular type of wind profile and have good agreement with them. In order to facilitate the application of the model the results for the general situation that includes chemical reaction rate and time dependent source incorporated in the model.**

**Key words:** Stable boundary layer, instantaneous line source, continuous line source, step-function type sources, chemically reactive pollutants, quadratic diffusion coefficient, and generalized parabolic wind profile.

## INTRODUCTION

The concentration of the pollutant in the atmosphere is generally influenced by several processes such as advection, turbulent diffusion, deposition, removal process and conversion of gaseous pollutants to the particulate materials. Pollutants may be removed from the atmosphere by natural cleansing processes e.g. washout/rainout and gravitational settling.

In this article we study the unsteady state dispersion of the chemically reactive pollutants into stable atmospheric boundary layer. K-theory might be most successful in the stable atmospheric boundary layer, where the diffusion coefficient is assumed a parabolic height dependence (Nokes et al., 1984, Neustadt, 1980). but quadratic profiles has also recently been proposed (Neustadt, 1984). In this model attention is focused on the generalized quadratic form of wind profile and pollutants are of chemically reactive primary pollutants. The steady state two dimensional model with non reactive gaseous pollutants from an elevated line source without settling has been studied by the earlier authors (Robson, 1987, Nokes et al., 1984, Neustadt, 1980, Neustadt, 1984). In this article the mathematical model for time-dependent and chemically reactive pollutants is developed. And moreover, the analytical solution of the model meets challenges over the numerical works had done by the authors in previous works.

### Formulation of the Mathematical Model

The dispersion of chemically reactive primary admixture's concentration in a turbulent atmospheric medium, when admixture gravitational precipitation cannot be neglected, is actually described by the following diffusion equation. The following mass transform equation describes the dynamics of advection, diffusion of chemically reactive pollutants.

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$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=\frac{\partial}{\partial x}\left(K_x\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial C}{\partial z}\right)-K''C+S \quad 2.1$$

Here $t$ is the time, $x, y, z$ are the space co-ordinates with z-axis along the vertical, $C$ the concentration of primary pollutant, $(u, v, w)$ the component of wind in $x, y, z$ direction respectively and $K_x, K_y$, $K_z$ are the turbulent diffusion coefficients along $x, y, z$ -directions respectively. $K''$ is the chemical reaction rate and $S$ is the source. We assume that the pollutants are emitting from the elevated line source of time dependent. The model is of time-dependent. In addition the following assumptions are made according to the physical situation concern to the model.

(i) Convection dominates over horizontal diffusion (x-direction) and vertical diffusion dominates over advection.

$$u\frac{\partial C}{\partial x} >> \frac{\partial}{\partial x}\left(K_x\frac{\partial C}{\partial x}\right) \text{ and}$$

$$w\frac{\partial C}{\partial z} << \frac{\partial}{\partial z}\left(K_z\frac{\partial C}{\partial z}\right)$$

(ii) The wind direction is assumed to be in the x-direction and hence the velocity component along y-axis is zero.

$v = 0$

(iii) The elevated line source is in the direction of y-axis, therefore, the flux gradient of concentration is zero in the y-direction

$$v\frac{\partial C}{\partial y} = 0 \quad \& \quad \frac{\partial}{\partial y}\left(K_y\frac{\partial C}{\partial y}\right) = 0$$

(iv) The source $S$ is taken to be of time dependent line source and it is assumed to be in the y-direction i.e.

$$S = QW(t)\delta(x-x_o)\delta(z-z_o) \quad 2.2$$

where $(x_o, z_o)$ denotes the co-ordinates of source.

The possible forms of the source terms are:

a. An instantaneous type of source:

$$W(t) = W_o\delta(t) \quad 2.3$$

b. The continuous type of source:

$$W(t) = W_c \quad 2.4$$

c. Step-Function type of source:

$$W(t) = \begin{cases} W_C, & 0 < t \le t_o \\ 0, & t > t_o \end{cases} \quad 2.5$$

(v) The reaction rate is assumed to be of the first order decay.

Based on the above assumptions the transport and diffusion equation (2.1) becomes:

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}=\frac{\partial}{\partial z}\left(K_z\frac{\partial C}{\partial z}\right)-K''C+QW(t)\delta(x-x_0)\delta(z-z_0) \quad 2.6$$

Initial and boundary conditions are:

$$C(x,z,t)=0 \quad at\ t=0, \qquad \forall\, x,\ z \in R^+ \quad 2.7$$

$$C(x,z,t)=0 \quad at\ x=0, \qquad t \ge 0, \qquad 0 \le z < H \quad 2.8$$

$$K_z\frac{\partial C(x,z,t)}{\partial z}=0, \qquad z = H, \ \forall \ x,t \ge 0 \quad 2.9$$

$$K_z\frac{\partial C(x,z,t)}{\partial z}=0, \qquad z = 0, \ \forall \ x,t \ge 0 \quad 2.10$$

**Meteorological parameters**

In a stable atmospheric boundary layer the diffusion coefficient $K_z$ and wind velocity $u(z)$ are taken to be functions of height –z as below;

$$u(z) = \frac{u_*}{k}\left\{\ln\left(\frac{z}{z_o}\right)+4.7\frac{z}{L}\right\} \quad 3.1$$

$$K_z = \frac{ku_* z(1-z/H)^2}{1+4.7\,z/L} \quad 3.2$$

In an atmospheric boundary layer we may distinguish the surface layer and the outer layer. In the surface layer the wind speed profile behaves like lnz and exchange coefficient behaves like z. this behavior can be recognized in the expression given above. If, however, we make the assumption that the surface layer is small with respect to the boundary-layer height and that as a first approximation the effect of the surface layer on the dispersion of, for example, an elevated source may be neglected, we can simplify the u(z) and K(z) profile to

$$u(z) = 4.7\frac{u_*}{k}\frac{z}{L} \quad 3.3$$

$$K_z = \frac{ku_*}{4.7} L(1 - z/H)^2 \qquad 3.4$$

But in this article we assumed the wind profile and exchange coefficient profiles in the power series and parabolic forms respectively. In 3.1 and 3.2 $u_*$ is the friction velocity, *L* the Monin-Obukov length, *k* von Karman constant and *H* the boundary layer height.

**Dimensional Analysis**

The analysis is facilitated through the introduction of the dimensional variables. Thus we have

$$Z = \frac{z}{H}, \qquad \frac{K_z}{K_o} = (1-Z)^2, \quad X = \frac{K_o}{H^2 u_o} x, \quad \frac{u(z)}{u_o} = \left[\frac{z}{H}\right]^n = Z^n,$$

Where, *n* is the constant decided on the stability of boundary layer. In the model specified by (2.6) the velocity profile is assumed in the generalized form of the powers type:

$$\frac{u(z)}{u_o} = a_o + a_1 Z + a_2 Z^2 \qquad 4.1$$

$$t' = \frac{K_o}{H^2} t, \qquad c' = \frac{H<u>}{W_o Q} c,$$

$$\text{where, } <u> = \frac{u_o}{n+1}$$

$$\text{and} \quad u_o = \frac{4.7 u_* H}{kL}, K_o = \frac{k}{4.7} u_* L \qquad \text{(Robson R.E-1987}$$

**Dimensional form of source terms**

i. $W(T) = \dfrac{W(t)}{W_o} = \delta(T)$ Impulse source 4.2

ii. $W(T) = \dfrac{W(t)}{W_C} = 1$ Continuous source 4.3

iii. $W(T) = \dfrac{W(t)}{W_C} = \begin{cases} 1, & 0 \le T \le T_o \\ 0, & T > T_o \end{cases}$ Step-Function type source. 4.4

Incorporating all the non-dimensional variables in the equation (2.6) we get:

$$\frac{W_o K_o}{H<u>H^2}\frac{\partial C}{\partial T} + \frac{W_o K_o}{H<u>u_o}\frac{u_o}{H^2}\left(a_o + a_1 Z + a_2 Z^2\right)\frac{\partial C}{\partial X} =$$

$$\frac{W_o K_o}{H<u>H^2}\frac{\partial}{\partial Z}\left((1-Z)^2 \frac{\partial C}{\partial Z}\right) - \frac{W_o K''}{H<u>} C + \frac{W_o W(T)}{H H^2 u_o}\delta(X - X_O)\delta(Z - Z_O)$$

4.5

$$\frac{\partial c}{\partial T} + \left(a_o + a_1 Z + a_2 Z^2\right)\frac{\partial c}{\partial X} =$$

$$\frac{\partial}{\partial Z}\left((1-Z)^2 \frac{\partial c}{\partial Z}\right) - \alpha c + \frac{W(T)}{\rho}\delta(X - X_O)\delta(Z - Z_O)$$

4.6

Dropping out the capitals the above differential equation with initial and boundary conditions becomes.

$$\frac{\partial c}{\partial t} + \left(a_o + a_1 z + a_2 z^2\right)\frac{\partial c}{\partial x} =$$

$$\frac{\partial}{\partial z}\left((1-z)^2 \frac{\partial c}{\partial z}\right) - \alpha c + \frac{W(t)}{\rho}\delta(x - x_O)\delta(z - z_O) \qquad 4.7$$

Where,

$$\rho = a_0 + 2a_1 + 3a_2 \qquad 4.8$$

$$\alpha = \frac{K'' H^2}{K_o}$$ (non-dimensional reaction rate parameter).

4.9

Initial and boundary conditions are

$$c(x,z,t) = 0 \quad at\ t = 0, \qquad \forall\, x,\ z \in R^+ \qquad 4.8$$

$$c(x,z,t) = 0 \quad at\ x = 0, \qquad t = 0, \quad 0 \le z < 1 \qquad 4.9$$

$$(1-z)^2 \frac{\partial c(x,z,t)}{\partial z} = 0, \qquad z = 1\ , \ \forall \quad x,t \ge 0 \qquad 4.10$$

$$(1-z)^2 \frac{\partial c(x,z,t)}{\partial z} = 0, \qquad z = 0\ , \ \forall \quad x,t \ge 0$$

4.11

**Method of Solution**

Taking Laplace transform along *t*, the partial differential equation (4.7) reduced to the ordinary differential equation as below

$$s\bar{c}\,(x,z,s)-c\,(z,0)+\left(a_o+a_1z+a_2z^2\right)\frac{\partial\bar{c}}{\partial x}=\frac{\partial}{\partial z}\left((1-z)^2\frac{\partial\bar{c}}{\partial z}\right)-\alpha\bar{c}+\frac{\overline{W(s)}}{\rho}\delta(x-x_0)\delta(z-z_0) \quad 5.1$$

or

$$s\bar{c}\,(x,z,s)-c\,(x,0)+\left(\sum_0^2 a_iz^i\right)\frac{\partial\bar{c}}{\partial x}=\frac{\partial}{\partial z}\left((1-z)^2\frac{\partial\bar{c}}{\partial z}\right)-\alpha\bar{c}+\frac{\overline{W(s)}}{\rho}\delta(x-x_0)\delta(z-z_0) \quad 5.2$$

$$\left(\sum_0^2 a_iz^i\right)\frac{\partial\bar{c}}{\partial x}=\frac{\partial}{\partial z}\left((1-z)^2\frac{\partial\bar{c}}{\partial z}\right)-(\alpha+s)\bar{c}(x,z,s)+\frac{\overline{W(s)}}{\rho}\delta(x-x_0)\delta(z-z_0) \quad 5.3$$

Where,

$$\bar{c}\;(x,z,s)=\int_0^\infty e^{-st}c(x,z,t)dt \quad 5.4$$

This is a 2-Dimensional partial differential equation along with the initial and boundary conditions:

$$\bar{c}(x,z,s)=0 \quad at\ x=0, \qquad 0\le z<1 \quad 5.5$$

$$(1-z)^2\frac{\partial\bar{c}(x,z,s)}{\partial z}=0, \qquad z=1\,,\ \forall\ \ x\ge 0 \quad 5.6$$

$$(1-z)^2\frac{\partial\bar{c}(x,z,s)}{\partial z}=0, \qquad z=0\,,\ \forall\ \ x\ge 0 \quad 5.7$$

Again applying the Laplace transform along x-variable the above 2-D partial differential equation reduced to the ordinary differential equation with only boundary conditions:

$$\frac{d}{dz}\left((1-z)^2\frac{d\hat{c}}{dz}\right)-(\alpha+s+p\left(\sum_0^2 a_iz^i\right))\hat{c} \;=\; -\frac{\overline{W(s)}}{\rho}e^{-px_o}\delta(z-z_0) \quad 5.8$$

$$(1-z)^2\ \frac{d\hat{c}}{dz}=0, \qquad at\quad z=1 \quad 5.9$$

$$(1-z)^2\ \frac{d\hat{c}}{dz}=0, \qquad at\quad z=0 \quad 5.10$$

$$\hat{c}\;(p,z,s)=\int_0^\infty e^{-pt}\,\bar{c}(x,z,s)dx \quad 5.11$$

The solution of the above equation can be obtained via Green’s Function technique in two regions for $z<z_o$ (region below the source) and for $z>z_o$ (region above the source) and up to the inverse layer whose height is 1(non-dimensional parameter of height).

**The reasons for using the Green’s Function technique are**

(i) Green’s Functions are flexible and powerful. The same GF for a given geometry and given set of homogeneous boundary conditions is building block for concentration distribution resulting from (a) space variable initial concentration distribution, (b) time-and space-variable boundary conditions and (c) time and space variable volume energy generation.
(ii) a second advantage of GF method is the systematic solution procedure. The saving effort and reduced possibility of errors are particularly important for two-and three dimensional geometries.
(iii) a third advantage is that 2 and3 dimensional GF’s can be found by simple multiplication of one dimensional GF’s for the rectangular coordinate system.
(iv) a fourth advantage is that the GF solution equation has an alternative form which can improve the convergence of problems with non-homogeneous boundary conditions.
(v) a fifth advantage is that the GF solution method can be time partitioned to reduce the no. of terms of an infinite series that must be evaluated. Time partitioned is a general method. The method of time partitioning can give accurate values for the concentration using only few terms in the infinite series.

**Case 1**

**When the source is of impulse type**:

When, the source is of impulse type $W(t)=\delta(t)$ then, $\overline{W(s)}=1$. The solution of the equation (5.8) may be obtained through Green’s Function method as

$$\hat{C}(p,z,s)=\begin{cases} -\dfrac{e^{-px_0}\phi_a(z)\phi_b(z_o)}{\rho A_1}, & z<z_o \\[2ex] -\dfrac{e^{-px_0}\phi_a(z_o)\phi_b(z)}{\rho A_1}, & z>z_o \end{cases} \quad 5.12$$

Where, $\phi_a(z)$ and $\phi_b(z)$ are the two linearly independent solutions of the homogeneous differential equation

$$\frac{d}{dz}\left((1\text{-}z)^2\ \frac{d\phi}{dz}\right)-\left(\alpha+s+p\left(\sum_0^2 a_i z^i\right)\right)\phi=0$$ 5.13

satisfying the boundary conditions:

$$(1\text{-}z)^2\ \frac{d\phi}{dz}=0, \qquad at \quad z=1$$ 5.14

$$(1\text{-}z)^2\ \frac{d\phi}{dz}=0, \qquad at \quad z=0$$ 5.15

A is determined by the Wronskian

$$W[\phi,\phi]_{z=z_o}=\phi_a(z_o)\phi_b'(z_o)-\phi_a'(z_o)\phi_b(z_o)=\frac{A_1}{(1-z_o)^2}$$ 5.16

Now introducing a new type of transformation in equation (5.13):

$\eta=2\sqrt{a_2 p}(1-z)$ we obtain 5.17

$$\frac{d}{dz}=\frac{d}{d\eta}\frac{d\eta}{dz}=-2\sqrt{pa_2}\frac{d}{d\eta}$$ 5.18

$$\frac{d^2}{dz^2}=\frac{d^2}{d\eta^2}\left(\frac{d\eta}{dz}=\right)^2 4pa_2\frac{d^2}{d\eta^2}$$ 5.19

The homogeneous equation (5.13) becomes

$$\eta^2\frac{d^2\phi}{d\eta^2}+2\eta\frac{d\phi}{d\eta}-\left\{\alpha+s+p(a_o+a_1+a_2)-p\left(\frac{a_1+2a_2}{2\sqrt{pa_2}}\right)\eta+\frac{\eta^2}{4}\right\}\phi=0$$ 5.20

or

$$\eta^2\frac{d^2\phi}{d\eta^2}+2\eta\frac{d\phi}{d\eta}+\left\{-\frac{\eta^2}{4}+\lambda_1\eta-\lambda_2\right\}\phi=0,$$ 5.21

Where,

$$\lambda_1=\left(\frac{a_1+2a_2}{2\sqrt{a_2}}\right)\sqrt{p}$$ 5.22

and

$$\lambda_2=\alpha+s+p(a_o+a_1+a_2).$$ 5.23

Let' assume the transformation

$$\phi=\frac{1}{\eta}W(\eta)$$ 5.24

where $\phi$ is the solution of the differential equation (5.21). The equation (5.21) becomes after introducing the above transformation:

$$\frac{d^2W}{d\eta^2}+\left\{-\frac{1}{4}+\frac{\lambda_1}{\eta}-\frac{\lambda_2}{\eta^2}\right\}W=0$$ 5.25

General form of the **Whittaker differential** equation as fallows, by comparing the parameters of equation 5.25

$$\frac{d^2W}{d\eta^2}+\left\{-\frac{1}{4}+\frac{\kappa}{\eta}+\frac{1/4-m^2}{\eta^2}\right\}W=0$$ 5.26

Where,

$\kappa=\lambda_1$ 5.27

Whose, solution is $W_{\kappa,\pm m}(\eta)$

$$\phi_a(z)=\frac{W_{k,m}(\eta)+R_1W_{k,-m}(\eta)}{\eta}$$ 5.28

$$\phi_b=\frac{1}{\eta}W_{k,m}(\eta)$$ 5.29

$$W_{k,m}(\eta)=\eta^{\frac{1}{2}+m}e^{-\frac{\eta}{2}}F(\frac{1}{2}-k+m,1+2m,\eta)$$ 5.30

$$W_{k,-m}(\eta)=\eta^{\frac{1}{2}-m}e^{-\frac{\eta}{2}}F(\frac{1}{2}-k-m,1-2m,\eta)$$ 5.31

$$F(\alpha,\beta,\frac{a^2}{4t})=e^{\frac{a^2}{4t}}(\alpha-\beta)!(\beta-1)!\left(\frac{a}{2}\right)^{1-\beta}.t^{\beta-\alpha}.p^{\frac{\beta}{2}-\alpha+\frac{1}{2}}J_{\beta-1}\left(ap^{1/2}\right)$$ 5.32

$$F(\alpha,\beta,\frac{a^2}{4t})=\sum_0^\infty\frac{(\alpha)_j}{j!(\beta)_j}$$ 5.33

$$p^n=\frac{t^{-n}}{(-n)!}$$ 5.34

$$t^n=\frac{p^{-n}}{(-n)!}$$ 5.35

$$t^{\beta-\alpha}=\frac{p^{\alpha-\beta}}{(\alpha-\beta)!}$$ 5.36

$$(\alpha-\beta)!t^{\beta-\alpha}p^{\beta/2-\alpha+1/2}=p^{1/2(1-\beta)}$$

5.37

Therefore,

$$F\left(\frac{1}{2}-\lambda_1+m,1+2m,\eta\right)=\frac{(2m)!}{m!}\frac{e^{\eta}}{\eta^{m}}J_{2m}\left(2\sqrt{\frac{\eta}{\pi}}\right)$$

5.38

Equations 5.30 and 5.31 becomes,

$$W_{k,m}(\eta)=\eta^{\frac{1}{2}}e^{\frac{\eta}{2}}J_{2m}\left(2\sqrt{\frac{\eta}{\pi}}\right) \qquad 5.39$$

$$W_{k,-m}(\eta)=\eta^{\frac{1}{2}}e^{\frac{\eta}{2}}J_{-2m}\left(2\sqrt{\frac{\eta}{\pi}}\right) \qquad 5.40$$

These are the two independent solution of the homogeneous differential equation (5.21) however, $2m$ is assumed to be not an integer. Therefore, we choose the linearly independent solutions of the equation as:

$$\phi_a(z)=\frac{W_{k,-m}(\eta)+R_1W_{k,m}(\eta)}{\eta} \qquad 5.41$$

$$\phi_b(z)=\frac{W_{k,m}(\eta)}{\eta} \qquad 5.42$$

$$\phi_a(z)=\eta^{-\frac{1}{2}}e^{\frac{\eta}{2}}J_{-2m}\left(2\sqrt{\frac{\eta}{\pi}}\right)+R_1\eta^{-\frac{1}{2}}e^{\frac{\eta}{2}}J_{2m}\left(2\sqrt{\frac{\eta}{\pi}}\right)$$

5.43

$$\phi_b(z)=\eta^{-\frac{1}{2}}e^{\frac{\eta}{2}}J_{2m}\left(2\sqrt{\frac{\eta}{\pi}}\right) \qquad 5.44$$

$$\phi_a(\eta)=\frac{e^{\eta/2}}{\sqrt{\eta}}\left[J_{-2m}\left(2\sqrt{\frac{\eta}{\pi}}\right)+R_1J_{2m}\left(2\sqrt{\frac{\eta}{\pi}}\right)\right]$$

5.45

$$\phi_b(\eta)=\frac{e^{\eta/2}}{\sqrt{\eta}}J_{2m}\left(2\sqrt{\frac{\eta}{\pi}}\right) \qquad 5.46$$

Approximating, **Bessel's**

**function** $J_{2m}\left(2\sqrt{\frac{\eta}{\pi}}\right)\cong\left(\sqrt{\frac{\eta}{\pi}}\right)^{2m}$ and

$J_{-2m}\left(2\sqrt{\frac{\eta}{\pi}}\right)\cong\left(\sqrt{\frac{\eta}{\pi}}\right)^{-2m}$ for small values of '$\eta$'i.e. small $p\to 0$ or $x\to\infty$ (Beck et al, 1992).

Where, $R_1$ can be determined by using the boundary conditions, and is found to be

$$R_1=-\left(\frac{\pi^2}{4p}\right)^{m}\left[\frac{\sqrt{a_2p}-m-1/2}{\sqrt{a_2p}+m-1/2}\right] \qquad 5.47$$

and $A_1$ is determined as

$$A_1=-4m(1-z_o)^2\sqrt{a_2p}e^{\sqrt{a_2p}(1-z_o)} \qquad 5.48$$

Then, the GF's solution (5.12) becomes,

$$\hat{c}_1(p,z,s)=\frac{e^{z_o\sqrt{a_2p}}}{2\rho\sqrt{(1-z_o)^3}}\left[\frac{e^{-mh_1}}{m}+R_1\left(\frac{4p}{\pi^2}\right)^{m}\frac{e^{-mh_2}}{m}\right]$$

5.49

$$\hat{c}_1(p,z,s)=\frac{e^{z_o\sqrt{a_2p}}}{2\rho\sqrt{(1-z_o)^3}}\left[\frac{e^{-mh_1}}{m}-\left(\frac{\sqrt{a_2p}-m-1/2}{\sqrt{a_2p}+m-1/2}\right)\frac{e^{-mh_2}}{m}\right]$$

5.50

Assuming z = 0, i.e. the concentration at ground level which practically necessary to assess the impact of pollutant. Then

$$h_1=h_2=\log\left[\frac{1}{1-z_o}\right]=h\,(say) \qquad 5.51$$

Then the equation (5.49) becomes

$$\hat{c}_1(p,0,s)=\frac{e^{z_o\sqrt{a_2p}}}{2\rho\sqrt{(1-z_o)^3}}\left[\frac{e^{-mh}}{\sqrt{a_2p}+m-1/2}\right], \qquad z<z_o$$

5.52

Using standard tables, Erdelyi et. all (1954), Schaum Series (1950) Beck et.all (1992), performing the multiple-inverse Lap lace transform over equation (8.5.51) and obtain the solution as:

$$C_1(x,0,t)=L_p^{-1}\left\{L_s^{-1}\left(\frac{e^{z_o\sqrt{a_2p}}}{6\sqrt{(1-z_o)}}\left[\frac{e^{-mh}}{\sqrt{a_2p}+m-1/2}\right]\right)\right\},\ z<z_o$$

5.53

Where,

$$m=\sqrt{\frac{1}{4}+\alpha+s+p(a_o+a_1+a_2)} \qquad 5.54$$

$$c_1(x,0,t)=\frac{he^{-\left(\frac{1}{4}+\alpha\right)t-\frac{h^2}{4t}}}{2\rho\sqrt{\pi(1-z_o)t^3a_a}}\left[\frac{e^{-\frac{a_2z_o^2}{4x}}}{\sqrt{4x}}-\left(\frac{h}{2t\sqrt{a_2}}+1/2\right)\Psi_1(z,t)\Psi_2(z,t)\right],\quad z_o>0$$

5.55

$$\Psi_1(z,t)=e^{-z_o\left(\frac{h}{2t}+\frac{\sqrt{a_2}}{2}\right)+\left(\frac{h^2}{4a_2t^2}+1/4+\frac{h}{2t\sqrt{a_2}}\right)x} \quad 5.56$$

$$\Psi_2(z,t)=Erfc\left[\left(\frac{h}{2t\sqrt{a_2}}+1/2\right)\sqrt{x}-\frac{z_o}{2}\sqrt{\frac{a_2}{x}}\right] \quad 5.57$$

$$\rho=a_o+2a_1+3a_2 \quad 5.58$$

$$h=\log\left(\frac{1}{1-z_o}\right) \quad 5.59$$

**Case 2**

**When the source is of Step-Function type**

If the source is of Step-Function type, $U(t-t_o)=\begin{cases}1, & 0<t\le t_o\\ 0, & t>t_o\end{cases}$ then the source is switched on during a fixed period of time and then switched off. The equation (5.8) and along with boundary conditions becomes:

$$\frac{d}{dz}\left((1\text{-}z)^2\frac{d\hat{c}}{dz}\right)-(\alpha+s+p\left(\sum_0^2 a_iz^i\right))\hat{c}=-\frac{\overline{W(s)}}{\rho}e^{-px_0}\delta(z-z_0)$$

5.60

$$(1\text{-}z)^2\frac{d\hat{c}}{dz}=0,\qquad at\quad z=1 \quad 5.61$$

$$(1\text{-}z)^2\frac{d\hat{c}}{dz}=0,\qquad at\quad z=0 \quad 5.62$$

Where,

$$\overline{W(s)}=\frac{1-e^{-st_o}}{s} \quad 5.63$$

Solutions of ordinary differential equation (5.60) through **Green's Function** technique at z= 0 becomes:

$$\hat{c}_1(p,0,s)=\overline{W(s)}\frac{e^{z_o\sqrt{a_2p}}}{2\rho\sqrt{(1-z_o)^3}}\left[\frac{e^{-mh}}{m}-\left(\frac{\sqrt{a_2p}-m+1/2}{\sqrt{a_2p}+m+1/2}\right)\frac{e^{-mh}}{m}\right],\quad z<z_0$$

5.64

Where,

$$m=\sqrt{\frac{1}{4}+\alpha+s+p(a_o+a_1+a_2)} \quad 5.65$$

$$\hat{c}_1(p,0,s)=\overline{W(s)}\frac{e^{z_o\sqrt{a_2p}}}{\rho\sqrt{(1-z_o)^3}}\left[\frac{e^{-mh}}{\sqrt{a_2p}+m+1/2}\right],\qquad z<z_0$$

5.66

Taking inverse lap lace transform along s, we get

$$\overline{C}_1(p,0,t)=\frac{e^{z_o\sqrt{a_2p}}}{\rho\sqrt{(1-z_o)^3}}L_s^{-1}\left[\frac{\overline{W(s)}e^{-mh}}{\sqrt{a_2p}+m+1/2}\right],\qquad z<z_0$$

5.67

$$\overline{C}_1(p,0,t)=\frac{e^{z_o\sqrt{a_2p}}}{\rho\sqrt{(1-z_o)^3}}\left\{\int_0^t W(u)E(t-u)du\right\},\qquad z<z_0$$

5.68

$$L_s^{-1}\left\{\overline{W(s)}\right\}=W(t)=U(t-t_o) \quad 5.69$$

$$E(t)=L_s^{-1}\left[\frac{\overline{W(s)}e^{-mh}}{V+\sqrt{a_2p}+m}\right] \quad 5.70$$

$$E(t)=e^{-\left(\frac{1}{4}+\alpha+p\sum_0^2 a_j\right)t}\left\{\frac{e^{-\frac{h^2}{4t}}}{\sqrt{\pi t}}-\beta_1e^{h\beta_1+\beta_1^2t}Erfc\left(\frac{h}{2\sqrt{t}}+\beta_1\sqrt{t}\right)\right\}$$

5.70

$$E(t)=e^{-\beta_2^2t}\left\{\frac{e^{-\frac{h^2}{4t}}}{\sqrt{\pi t}}-\beta_1e^{h\beta_1+\beta_1^2t}Erfc\left(\frac{h}{2\sqrt{t}}+\beta_1\sqrt{t}\right)\right\}$$

5.71

$$\beta_1=1/2+\sqrt{a_2p} \quad 5.72$$

$$\beta_2 = \sqrt{\frac{1}{4} + \alpha + p\sum_{0}^{2} a_j} \quad 5.73$$

$$h = \log\left[\frac{1}{1 - z_o}\right] \quad 5.74$$

Equation (5.68) becomes

$$\overline{C}(p,0,t) = \frac{e^{z_o\sqrt{a_2 p}}}{\rho\sqrt{(1-z_o)^3}}\left\{\int_0^{t_o} W(u)E(t-u)du + \int_{t_o}^{t} W(u)E(t-u)du\right\}$$

$$= \frac{e^{z_o\sqrt{a_2 p}}}{\rho\sqrt{(1-z_o)^3}}\left\{\int_0^{t_o} W(u)E(t-u)du\right\} \qquad z < z_0$$

$$= \frac{e^{z_o\sqrt{a_2 p}}}{\rho\sqrt{(1-z_o)^3}}\left\{\int_{t-t_o}^{t} E(t)du\right\}$$

$$= \frac{e^{z_o\sqrt{a_2 p}}}{\rho\sqrt{(1-z_o)^3}}\int_{t-t_o}^{t} e^{-\beta_2^2 t}\left\{\frac{e^{-\frac{h^2}{4t}}}{\sqrt{\pi t}} - \beta_1 e^{h\beta_1 + \beta_1^2 t} Erfc\left(\frac{h}{2\sqrt{t}} + \beta_1\sqrt{t}\right)\right\}$$

$$\overline{C}(p,0,t) = \frac{-he^{z_o\sqrt{a_2 p}}}{2\rho\sqrt{\pi(1-z_o)^3}}\left[\phi(t) - \phi(t-t_o)\right] \quad 5.76$$

Where,

$$\phi(t) = \frac{e^{-\frac{h^2}{4t} - \beta_2^2 t}}{\left(\beta_2^2 t - \frac{h^2}{4t}\right)\left(\beta_1 t + \frac{h}{2\sqrt{t}}\right)} \quad 5.78$$

$$C_1(x,0,t) = \frac{-h}{2\rho\sqrt{\pi(1-z_o)^3}} L_p^{-1}\left\{e^{z_o\sqrt{a_2 p}}\left[\phi(t) - \phi(t-t_o)\right]\right\}$$

5.79

$$L_p^{-1}\left\{e^{z_o\sqrt{a_2 p}}\left[\phi(t)\right]\right\} = \int_0^x G_1(v,t)G_2(x-v,t)dv \qquad n$$

5.80

Where,

$$G_1(x,t) = e^{\lambda_3^2 x} U\left(x + t\sum_0^2 a_i\right) \quad 5.81$$

$$G_2(x,t) = \frac{1}{\sqrt{a_2 t}}\left\{\frac{e^{-\frac{\lambda_1^2}{4x}}}{\sqrt{\pi x}} - \lambda_2 e^{\lambda_1\lambda_2 + \lambda_2^2 t} Erfc\left(\frac{\lambda_1}{2\sqrt{t}} + \lambda_2\sqrt{t}\right)\right\}$$

5.82

$$L_p^{-1}\left\{e^{z_o\sqrt{a_2 p}}\left[\varphi(t)\right]\right\} = \frac{-e^{\lambda_3^2 x}}{2\sqrt{a_2 t}}\left[F_1(x) - F_2(x) - F_3(x)\right] = \psi_1(x,t)$$

5.83

Equation (5.79) becomes

$$C_1(x,0,t) = \frac{h}{2\rho\sqrt{\pi(1-z_o)^3}}\left\{\psi_1(x,t) - \psi_1(x,t-t_o)\right\}$$

5.84

Where.

$$\psi_1(x,t) = \frac{-e^{\lambda_3^2 x}}{2\sqrt{a_2 t}}\left[F_1(x,t) - F_2(x,t) - F_3(x,t)\right]$$

5.85

$$F_1(x,t) = \frac{\lambda_2 e^{\lambda_1\lambda_2 + \left(\lambda_2^2 - \lambda_3^2\right)\xi}}{\left(\lambda_2^2 - \lambda_3^2\right)} Erfc\left(\frac{\lambda_1}{2\sqrt{\xi}} + \lambda_2\sqrt{\xi}\right)$$

5.86

$$F_2(x,t) = e^{\lambda_1\lambda_3} Erfc\left(\frac{\lambda_1}{2\sqrt{\xi}} + \lambda_3\sqrt{\xi}\right) \quad 5.87$$

$$F_3(x,t) = e^{-\lambda_1\lambda_3} Erfc\left(\frac{\lambda_1}{2\sqrt{\xi}} - \lambda_3\sqrt{\xi}\right) \quad 5.88$$

$$\lambda_1 = -z_o\sqrt{a_2} \qquad 5.89$$

$$\lambda_2 = \frac{1/2 + h/2t}{\sqrt{a_2}} \qquad 5.90$$

$$\lambda_3^2 = \frac{\frac{h^2}{4t^2} - \left(\frac{1}{4} + \alpha\right)}{a_i} \qquad 5.91$$

$$\xi = x + a_i t \qquad 5.92$$

$$h = \log\left(\frac{1}{1 - z_o}\right) \qquad 5.93$$

$$a_i = a_o + a_1 + a_2 = \sum_0^2 a_i \qquad 5.94$$

**Case 3**

**When the source is of Continuous type**

When the source is of continuous elevated line source $W(t)=1$ then, $\overline{W(s)} = \frac{1}{s}$. The solution (5.52) of the differential equation (5.8) obtained through Green's Function method as by assuming $x_o = 0$ and $z = 0$.

$$\hat{c}_1(p,0,s) = \frac{e^{z_o\sqrt{a_2 p}}}{s\rho\sqrt{(1-z_o)^3}}\left[\frac{e^{-mh}}{1/2+\sqrt{a_2 p}+m}\right], \qquad z < z_o \qquad 5.95$$

$$\bar{c}_1(p,0,t) = \frac{e^{z_o\sqrt{a_2 p}}}{\rho\sqrt{(1-z_o)^3}}\int_0^t L_s^{-1}\left[\frac{e^{-mh}}{1/2+\sqrt{a_2 p}+m}\right]dt, \qquad z < z_o \qquad 5.96$$

Here,

$$L_s^{-1}\left[\frac{e^{-mh}}{1/2+\sqrt{a_2 p}+m}\right] = e^{-\left(\frac{1}{4}+\alpha+a_i P\right)t}\left\{\frac{e^{-\frac{h^2}{4t}}}{\sqrt{\pi t}} - \beta_1 e^{h\beta_1+\beta_1^2 t}\, erfc\left(\frac{h}{2t}+\beta_1\sqrt{t}\right)\right\} \qquad 5.97$$

$$\bar{c}_1(p,0,t) = \frac{h e^{z_o\sqrt{a_2 p}}}{2\rho\sqrt{\pi(1-z_o)^3}}\left\{\frac{e^{\frac{h^2}{4t}-\beta_2^2 t}}{\left(\frac{h^2}{4t}-\beta_2^2 t\right)\left(\frac{h}{2\sqrt{t}}-\beta_1\sqrt{t}\right)}\right\} \qquad z < z_o \qquad 5.98$$

Where,

$$\beta_1 = 1/2 + a_2\sqrt{p} \qquad 5.99a$$

$$\beta_2^2 = \frac{1}{4} + \alpha + a_i\sqrt{p} \qquad 5.99b$$

Now,

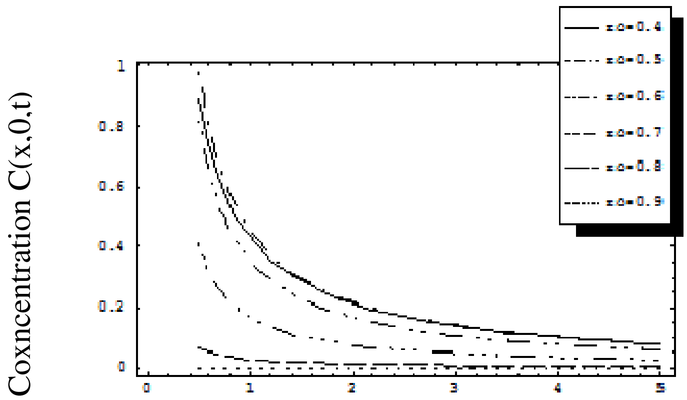
$$c_1(x,0,t) = \frac{h e^{-\frac{h^2}{4t}}}{2\rho\sqrt{\pi(1-z_o)^3}} L_p^{-1}\left\{\frac{e^{-\beta_2^2 t} e^{z_o\sqrt{a_2 p}}}{\left(\frac{h^2}{4t}-\beta_2^2 t\right)\left(\frac{h}{2\sqrt{t}}-\beta_1\sqrt{t}\right)}\right\}, \quad z < z_o \qquad 5.100$$

$$c_1(x,0,t) = \frac{h e^{-\frac{h^2}{4t}}}{2\rho\sqrt{\pi(1-z_o)^3}}\int_0^x M(x,t)N(x\text{-}u,t)du \qquad z < z_o \qquad 5.101$$

$$M(x,t) = L_p^{-1}\left\{\frac{e^{-\beta_2^2 t}}{\left(\frac{h^2}{4t}-\beta_2^2 t\right)}\right\} \qquad 5.102$$

$$\therefore M(x,t) = \frac{e^{(1/4+\alpha)\frac{x}{a_i}}}{a_i t}\begin{cases} -e^{\left(x-\frac{h^2}{4a_i t^2}\right)a_i t}, & x > a_i t \\ 0, & x < a_i t \end{cases} \qquad 5.103$$

$$N(x,t) = L_p^{-1}\left\{\frac{e^{z_o\sqrt{a_2 p}}}{\left(\frac{h}{2\sqrt{t}}-\beta_1\sqrt{t}\right)}\right\} \qquad 5.104$$




Distance (x)

**Figure. 1** Ground level (z = 0) concentration vs. source heights corresponding to the quadratic diffusion and parabolic wind coefficient, chemical reaction rate $\alpha = 0$ at time t= 0.1 for instantaneous Line Source.

$$\therefore N(x) = \frac{1}{a_2\sqrt{t}}\left\{\frac{e^{-\lambda_1^2/4x}}{\sqrt{\pi x}} - \lambda_2 e^{\lambda_1\lambda_2+\lambda_2^2 x}\, erfc\left(\frac{\lambda_1}{2\sqrt{x}} + \lambda_2\sqrt{x}\right)\right\}$$

5.105

The formula 5.100 becomes

$$c_1(x,0,t) = \frac{h e^{-\frac{h^2}{4t}-\lambda_3^2 x}}{2\rho\sqrt{\pi a_2 t(1-z_o)^3}}\{\varpi_1(x,t)-\varpi_2(x,t)-\varpi_3(x,t)\},\ z<z_o$$

5.106

where

$$\varpi_1(x,t) = \frac{\lambda_2 e^{\lambda_1\lambda_2}}{\left(\lambda_2^2-\lambda_3^2\right)} e^{\left(\lambda_2^2-\lambda_3^2\right)} erfc\left[\frac{\lambda_1}{2\sqrt{\xi}} + \lambda_2\sqrt{\xi}\right]$$

5.107(a)

$$\varpi_2(x,t) = \frac{e^{\lambda_1\lambda_2}}{2(\lambda_2-\lambda_3)} erfc\left[\frac{\lambda_1}{2\sqrt{\xi}} + \lambda_3\sqrt{\xi}\right]$$

5.107(b)

$$\varpi_3(x,t) = \frac{e^{-\lambda_1\lambda_2}}{2(\lambda_2+\lambda_3)} erfc\left[\frac{\lambda_1}{2\sqrt{\xi}} - \lambda_3\sqrt{\xi}\right]$$

5.107(c)

$$h = \log\left(\frac{1}{1-z_o}\right) \qquad 5.107(d)$$

$$\rho = a_o + 2a_1 + 3a_2 = \sum_0^2 (i+1)a_i \qquad 5.107(e)$$

$$\lambda_1 = -z_o\sqrt{a_2} \qquad 5.107(f)$$

$$\lambda_2 = \left(1/2 + \frac{h}{2t}\right)\Big/ a_2 \qquad 5.107(g)$$

$$\lambda_3 = \sqrt{(1/4+\alpha)\big/ a_i + a_i t} \qquad 5.107(h)$$

$$\xi = x - a_i t \qquad 5.107(i)$$

$$a_i = a_o + a_1 + a_2 = \sum_0^2 a_i \qquad 5.107(j)$$

## RESULTS AND DISCUSSIONS

In this paper, the dispersion of gaseous pollutants emitted from variable elevated line sources into a stable atmospheric boundary layer is simulated and studied by quadratic diffusion coefficient and generalized parabolic wind profile $u = a_o + a_1 z + a_2 z^2$. The three different line sources (Continuous, Impulse and Step-Function type of line sources) are incorporated to study the model. An analytical solution for all the above said cases found by multiple inverses of Laplace transforms through Green's Function technique. The constant wind and (n=0), but for constant shear wind profile (n=1) the numerical inversion of Laplace transform analyzed earlier (Robson, 1987). However, in this model the analytical solution for the generalized wind profile and for all the three different cases of various sources. The results are verified with the parabolic wind profile cases of the (Sulochana, 2009). The results are compared with the earlier work be C. (Sulochana and Shekhu, 2009, Robson 1987). These are of good agreement with that of the previous work by various authors. In the figure 1 above and 2 below.

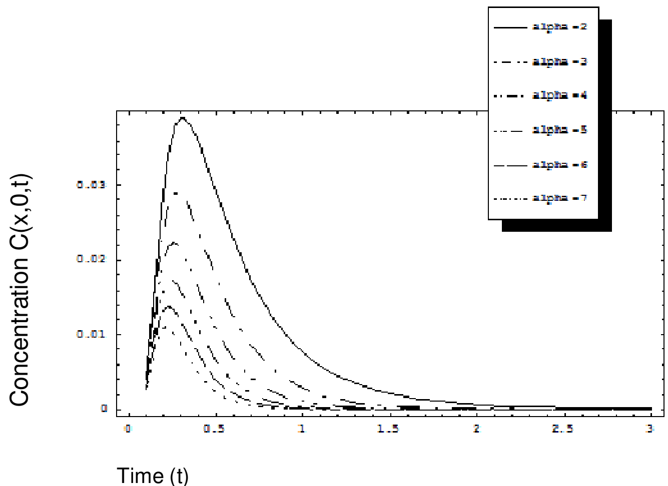



**Figure. 2** Ground level (z = 0) concentration vs. variable chemical reaction rate corresponding to the quadratic diffusion and wind coefficient of parabolic, source height $z_o = 0.8$ at Distance x = 4. for instantaneous Line Source.

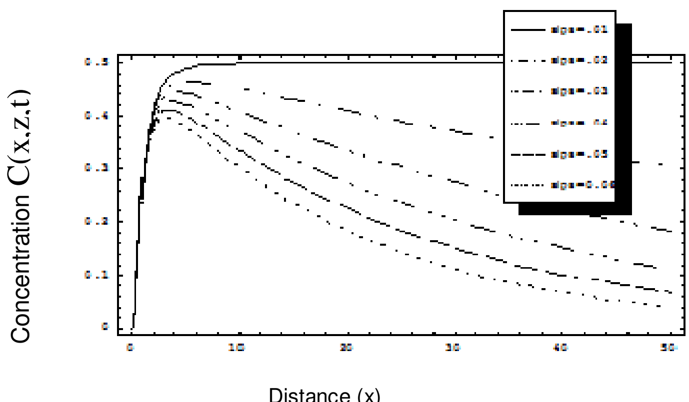



**Figure 3** Ground level (z=0) concentration for variable chemical reaction rate corresponding to the quadratic diffusion and wind velocity of constant shear, source height $z_o = 0.9$ for an instantaneous line source.

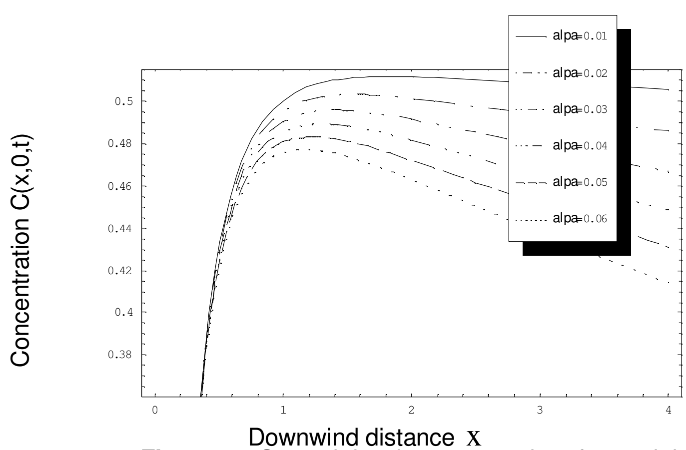



**Figure 4** Ground level concentration for variable chemical reaction rate corresponding to the quadratic diffusion and wind coefficient of constant shear, $z_o = 0.0$ step-function type elevated line source.

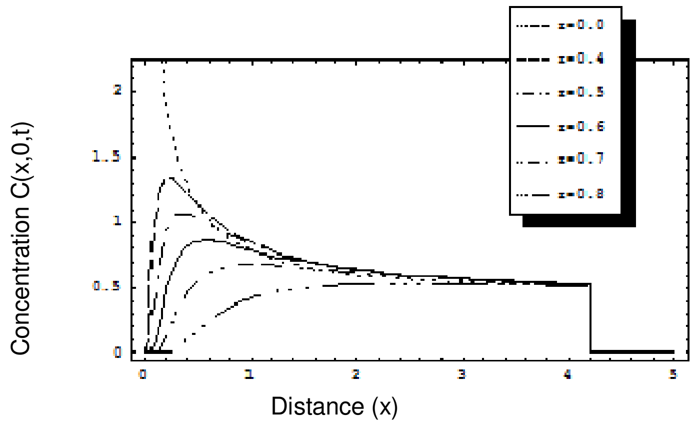



**Figure 5** Ground level concentration vs. Distance for various heights on the surface to the **quadratic** diffusion and constant shear wind coefficient, chemical reaction rate $\alpha = 0.0$ at time $t = 5, \quad t_o = 0.5$ step function type line source.

the results obtained by taking limiting case where $a_o, a_1 \to 0$ in wind profile considered in the present model are compared with that of the results in the previous work by (Sulochana and Shekhu, 2010). In the figure 3 above the results are validated by comparing the values for constant shear wind profiles for instantaneous line source by $a_o, a_2 \to 0$. In figures 4 and 5 above the results compared with that of the case where wind velocity is of constant shear and quadratic diffusion coefficient for step-function type line source. This obtained by taking the limiting case in the present model as $a_o, a_2 \to 0$.

**REFERENCE**

Alam MK, Seinfield JH (1981)*Solution of the steady state three dimensional diffusion equation for sulfur dioxide and sulfate dispersion from point source*. Atmos. Environ.15:1221-1225.

Beck JV, Cole KD, Haji-Sheikh, Litkouhi. (1992) '*Heat conduction Using Green's functions'*. Hemisphere Publishing Corporation, Washington, DC. pp-92-108

Donald L. Ermark. '*An analytical model for air pollutant transport and deposition from point source'*. Atmos. Env., 11(231-237):1976.

Sulochana C, Shekhu M (2009). Dispersion of Chemically Reactive Pollutants emitted from an Instantaneous Line source into a Stable Atmospheric Boundary Layer. *Pacific-Asian J. Mathematics*, 3(1-2): 103-117.

Sulochana C, Shekhu M (2009). Mathematical Model of Dispersion and Diffusion of Air pollutant from a Step-function type line source into a stable atmospheric boundary layer. *Indian J. Mathematics and Mathematical Sciences.* 5(2):125-143.

Uma A, Sunitha G, Gupta RS (1991) *Removal model suitable for air pollutants emitted from an elevated source*, Appl. Math. Modeling, 15: 386-389.

Erdelyi A, Magnus, Oberhettinger FG (1984) Tables *of Integral Transforms*, McGraw-Hill, New York. 1:321-328.

Hinderson B-Seller,(1984) *Pollution of Our Atmosphere*, University of Salford, Adam Hilger ltd, Bristol.Pp-97-108.

Kangle H, Martinus T, van Genuchten, renduo Z (1996), *Exact solutions for one-dimensional transport with asymptotic scale-dependent dispersion*. Appl. Math. Modeling, 20:298-308.

Kennith W, Cecil F, Warner (1981),*Air Pollution ( Its origin and Control)*, 2nd ed. Haipa and Raw. pp1576-1592.

Kennith W, Ragland, Robis L, Dennis (1975) *Point source atmospheric diffusion model with variable wind and diffusivity profiles*. Atmos. Environ, 19:175-189.

Nieuwstadt FTM (1984). *Some aspects of the turbulent stable boundary layer*, Boundary Layer Meteorology. 30:31-55.

Nieuwstadt FTM (1984). *The turbulent structure of the stable, nocturnal boundary layer*, J. Atmos. Sci. 41:2202–2216.

Nokes RI, Mcnulty AJ, Wood IR (1984). *Turbulent dispersion from a steady two-dimensional horizontal source*, J. Fluid Mechanics, 149: 147-159.

Rasmussen KH, Taheri M, Kabel RL (1975) *Global emission and natural processes for removal of gaseous pollutants*. Wat. Air and Soil pollut. 4: 33-64.

Robson RE (1987). '*Turbulent Dispersion in a Stable Layer with a Quadratic exchange Coefficient'* Boundary Layer Meteorology. 39: 207-218.

Robson RE (1987). *Turbulent dispersion in a stable layer with a quadratic exchange coefficient*, Boundary Layer Meteorology. 39: 207-218.