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Review

Exact solution of friction factor and diameter problems involving laminar flow of Bingham plastic fluids

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In this paper, mathematically equivalent representations of the *Buckingham-Reiner* equation to compute friction factor and of the diameter equation to compute circular pipe diameter for laminar flow of Bingham plastic fluids in pipes have been given. The new forms are simple and very well-suited for accurately estimating the friction factor and diameter, because no iterative calculations are necessary. Specifically, the friction factor and diameter are expressed as computable series using Lagrange's theorem. The equations presented in this study eliminate the need for best-fit parameters that are an integral part of the various explicit approximations proposed to date. The exact equations can also be utilized with advantage in optimization studies of pipelines. The ease, with which the new equations can be used, along with their smooth and predictable behavior, should make them the first choice of use for estimating the friction factor and diameter for laminar flow of Bingham plastic fluids in circular pipes.

Key Words: Bingham fluids, diameter, darcy-weisbach equation, explicit equation, friction factor, head losses, laminar flow.

INTRODUCTION

There are three typical problems encountered in pipe flows, depending upon what is known and what is to be found. These pertain to the determination of (1) friction factor, (2) discharge, and (3) pipe diameter (Darby 1996). The head loss in a pipe is given by

$$h_f = \frac{8fLQ^2}{\pi^2 gD^5} \qquad (1)$$

where f = Darcy-Weisbach friction factor, L = pipe length; Q = fluid discharge; g = gravitational acceleration; and D = pipe diameter. The continuity equation for discharge Q is

$$Q = \frac{\pi}{4} D^2 V \qquad (2)$$

where V = mean fluid velocity. Once the friction factor, f is known, it becomes easier to handle the different pipe-flow problems, for example calculating the pressure drop for evaluating pumping costs or finding the diameter of a pipe in a piping network for a given pressure drop and flow-rate. Due to the implicit nature of the friction factor equation, usually a large number of algebraic equations are generated for evaluating f when solving for fluid flow problems in a complex piping network (Brats etal., 1993).

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Notation

Following symbols have been used in this paper*:*

D pipe diameter; *f,* friction factor; *g,* gravitational acceleration; He, Hedstrom number; h*,* He/Re; h_f, frictional head loss; *L,* pipe length; *Q,* discharge; *q,* nondimensional discharge; Re, Reynolds number; S_f, friction slope; s_o, σ_0/ρ; *V,* average velocity; γ, velocity gradient; q, mass density; μ_∞, dynamic viscosity; v_∞, kinematic viscosity; σ, shear stress; σ_0, yield shear stress.

Subscript

$_*$, nondimensional.

This task becomes cumbersome as the pipe-network size increases and, in such cases, numerical approaches require a good initial guess of friction factor for each pipe element so that lesser computational efforts are needed. The same problem arises when the diameter of pipeline has to be obtained that carries a discharge Q with a given frictional loss h_f. To avoid this, researchers have tried to develop explicit approximations for the implicit friction factor equation for flow of Newtonian fluids (Swamee and Jain 1976, Haaland 1983, Serghides 1984, Goudar, and Sonnad 2007), and Non-Newtonian fluids (Darby 1996).

Bingham plastic fluid flow problems

The Bingham plastic model is given by the following rheological equation:

$$\tau = \tau_o + \rho \nu_\infty \gamma \qquad (3)$$

where τ = shear stress; τ_o = yield shear stress; ν_∞ = kinematic viscosity; ρ = mass density; and γ = velocity gradient perpendicular to the direction of shear. Many industrially important fluids, for e.g. various suspensions, slurries, drilling mud, pastes, gels, plastics, etc., exhibit non-Newtonian behavior and can be represented by the Bingham plastic model. Equations. (1)-(3) are quite useful in the design of pipelines for such fluids. These materials exhibit a yield stress which must be exceeded before they will flow at a significant rate. Other examples of such fluids include paint, shaving cream, and mayonnaise. There are also many "fluids" that may have a yield stress which is not as pronounced--- blood, for example. For laminar flow of such fluids, the friction factor, f, is given by the *Buckingham-Reiner* equation (Darby and Melson 1981)

$$f = \frac{64}{\mathrm{Re}}\left[1 + \frac{1}{6}\left(\frac{\mathrm{He}}{\mathrm{Re}}\right) - \frac{64}{3}\left(\frac{\mathrm{He}^4}{f^3\,\mathrm{Re}^7}\right)\right] \qquad (4)$$

where Re = Bingham Reynolds number; and He = Hedstrom number given by,

$$\mathrm{Re} = \frac{4Q}{\pi \nu_\infty D} \qquad (5)$$

$$\mathrm{He} = \frac{D^2 s_o}{\nu_\infty^2} \qquad (6)$$

where $s_o = \tau_o/\rho$; and Q = fluid discharge. The friction factor chart for laminar and turbulent flow conditions of Bingham plastic fluids is presented in Figure1. See Darby and Melson (1981). To solve for the friction factor, an exact analytical solution of the Buckingham-Reiner equation can be obtained as it involves a fourth order polynomial equation in f, but due to complexity of the solution it is not convenient. Recently some attempts have been made to obtain friction factor using the Buckingham-Reiner equation. For example see Sablani et al., 2003; Danish et al., 2011; Swamee and Aggarwal, 2011. For the diameter problem, since Re and He are unknown, it is also not possible to determine the friction factor using Equation (4) by straightforward trial and error procedure for subsequently finding diameter using Equation (1). Therefore, attempt for obtaining an explicit equation for diameter has been made recently. See Swamee and Aggarwal (2011). Similarly, in the discharge determination problem also, as Re and He are unknown, it is not possible to use straightforward trial and error procedure. Swamee and Aggarwal (2011) found an exact equation for discharge. The overall objective of this paper is to provide exact mathematical equations for friction factor and diameter from implicit Buckingham-Reiner equation.

Lagrange's theorem

Joseph Louis Lagrange (1736-1813) in 1770 gave a theorem by which solution of an implicit equation can be found in terms of an infinite series (Whittaker and Watson 1965; page 133). The theorem is stated as: A function g(y), where y is a root of the equation

$$y = a + \theta \phi(y) \qquad (7)$$

where a = constant; θ = parameter, and ϕ = function; is given by

$$g(y) = g(a) + \sum_{n=1}^{\infty} \frac{\theta^n}{n!}\left\{\frac{d^{n-1}}{dy^{n-1}}\left[g'(y)\phi^n(y)\right]\right\}_{y=a} \qquad (8)$$

where $g'(y) = dg(y)/dy$; and φ and g are differentiable functions at a (See for a generalized result, Whittaker and Watson 1965, page 133).

Friction factor problem

Eq. (4) is written in the two parameter form as

$$f\,\mathrm{Re} = 64 + \frac{32}{3}\frac{\mathrm{He}}{\mathrm{Re}} - \frac{4{,}096}{3(f\,\mathrm{Re})^3}\left(\frac{\mathrm{He}}{\mathrm{Re}}\right)^4 \qquad (9)$$

Small He/Re

Considering $g(f\mathrm{Re}) = f\mathrm{Re}$, $a = 64 + (32/3)(\mathrm{He}/\mathrm{Re})$; $\theta = -(4096/3)(\mathrm{He}/\mathrm{Re})^4$ and $\phi(f\mathrm{Re}) = 1/(f\mathrm{Re})^3$ the application of Lagrange theorem to Eq. (9) with some rearrangements gives f as

$$\frac{f\mathrm{Re}}{64\left(1 + \dfrac{\mathrm{He}}{6\,\mathrm{Re}}\right)} = 1 - \sum_{n=1}^{\infty}\left(\frac{27}{256}\right)^n \frac{(4n-2)!}{n!(3n-1)!P^{4n}} \qquad (10)$$

where

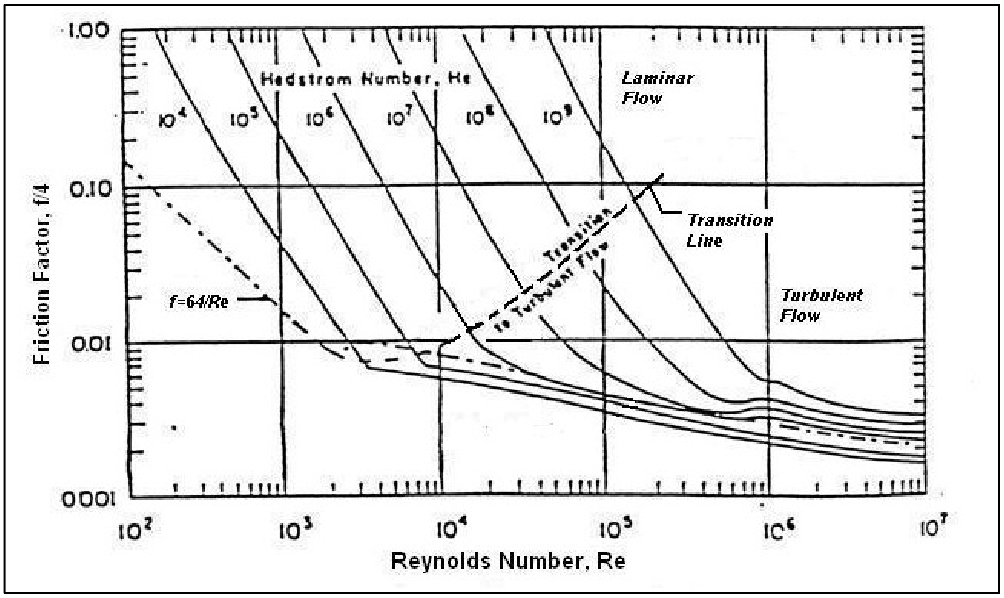

Figure 1. Friction factor for Bingham plastic fluids

$$P = \frac{6\,\mathrm{Re}}{\mathrm{He}} + 1 \qquad (11)$$

Expanding Eq. (10) and simplifying, the following series is obtained:

$$\frac{f\,\mathrm{Re}}{64\left(1+\frac{\mathrm{He}}{6\mathrm{Re}}\right)} = 1-\left(\frac{0.56988}{P}\right)^4-\left(\frac{0.65376}{P}\right)^8-\left(\frac{0.71415}{P}\right)^{12}-\left(\frac{0.75550}{P}\right)^{16}-\left(\frac{0.78545}{P}\right)^{20}\ldots$$
(12)

The series on the right hand side converges fast for large values of P, that is, when He/Re is small.

Large He/Re

Eq. (9) is written in the following implicit form:

$$\frac{\mathrm{He}}{\mathrm{Re}} = \left[\frac{f\,\mathrm{Re}^2}{64\mathrm{He}} + \frac{64}{3}\left(\frac{\mathrm{He}}{f\,\mathrm{Re}^2}\right)^3 - \frac{1}{6}\right]^{-1} \qquad (13)$$

Eq. (13) simplifies to

$$f\,\mathrm{Re}^2/\mathrm{He} = 8 + \left\{\frac{192\left(f\,\mathrm{Re}^2/\mathrm{He}\right)^3}{\left(\mathrm{He}/\mathrm{Re}\right)\left[3\left(f\,\mathrm{Re}^2/\mathrm{He}\right)^2+16f\,\mathrm{Re}^2/\mathrm{He}+64\right]}\right\}^{0.5} \qquad (14)$$

Applying Lagrange theorem to Eq. (14) by considering $g(f\mathrm{Re}^2/\mathrm{He}) = f\mathrm{Re}^2/\mathrm{He}$, $a = 8$; $\theta = (\mathrm{Re}/\mathrm{He})^{0.5}$ and

$$\phi\left(f\,\mathrm{Re}^2/\mathrm{He}\right) = \left\{192\left(f\,\mathrm{Re}^2/\mathrm{He}\right)^3 / \left[3\left(f\,\mathrm{Re}^2/\mathrm{He}\right)^2+16f\,\mathrm{Re}^2/\mathrm{He}+64\right]\right\}^{0.5},$$

$$f\,\mathrm{Re}^2/\mathrm{He} = 8+\left(\frac{256}{h}\right)^{\frac{1}{2}}+\frac{26.66667}{h}+\left(\frac{9.89184}{h}\right)^{\frac{3}{2}}+\left(\frac{4.07340}{h}\right)^2-\left(\frac{3.25977}{h}\right)^{\frac{5}{2}}+\ldots$$
(15)

where h = He/Re. For fast convergence of the series, h on the right side has to be large. Eqs. (12) and (15) are plotted in Figure (2). Errors involved in the Eqs. (12) and (15) are depicted in Figure 2. A perusal of Figure 2 reveals that the maximum error involved in the use of Eqs. (12) and (15) is about 0.013%. Thus, the series solution is almost exact. The maximum error occurs at about h = 30. Thus, Eq. (12) is valid for $\mathrm{He}/\mathrm{Re} \le 30$, whereas Eq. (15) is valid for $\mathrm{He}/\mathrm{Re} \ge 30$.

Diameter problem

Defining

$$q = \frac{\nu_\infty \left(g S_f\right)^3 Q}{s_o^4} \qquad (16)$$

$$D_* = \frac{g D S_f}{s_o} \qquad (17)$$

where $S_f = h_f/L$. Using Eqs. (1), (5), (16) and (17) one gets

$$f\,\mathrm{Re} = \frac{\pi D_*^4}{2q} \qquad (18)$$

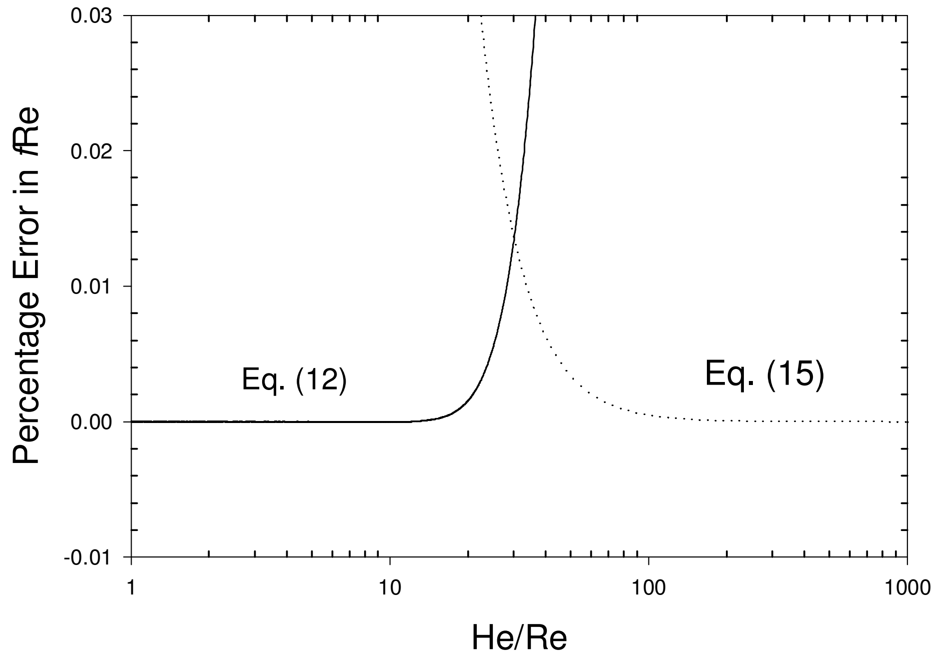

Figure 2. Percentage Errors in Eq. (12) and (15)

Similarly,using Eqs. (1), (5), (6), (16) and (17) one finds

$$\frac{\mathrm{He}}{\mathrm{Re}} = \frac{\pi D_*^3}{4q} \quad (19)$$

Using, Eqs. (18) and (19), Eq. (4) is written as

$$D_*^4 - \frac{16}{3}D_*^3 + \frac{256}{3}\left(1 - \frac{3q}{2\pi}\right) = 0 \quad (20)$$

Eq. (20) is rewritten as

$$D_*^3 = \frac{3}{16}D_*^4 + 16\left(1 - \frac{3q}{2\pi}\right) \quad (21)$$

Denoting $T = D_* - 4$, Eq. (21) is rewritten as

$$T^2\left(T^2 + \frac{32}{3}T + 32\right) = \frac{128q}{\pi} \quad (22)$$

For small q, taking $x = T^2$, $g(x) = x^{-1/2}$, $a = \pi/(4q)$,

$\theta = \pi/(128q)$ and $\phi(x) = x^{-1/2}(x^{-1/2}+32/3)$ in Eq. (22), and applying Lagrange theorem with some rearrangements gives D_* as

$$D_* = 4 + \left(\frac{q}{0.78540}\right)^{\frac{1}{2}} - \frac{q}{4.71239} + \left(\frac{q}{5.50973}\right)^{\frac{3}{2}} - \left(\frac{q}{5.36736}\right)^{2} + \left(\frac{q}{5.06622}\right)^{\frac{5}{2}} - \left(\frac{q}{4.77252}\right)^{3} + \left(\frac{q}{4.51764}\right)^{\frac{7}{2}} - \left(\frac{q}{4.30223}\right)^{4} + \left(\frac{q}{4.12051}\right)^{\frac{9}{2}} - \left(\frac{q}{3.96628}\right)^{5} + \left(\frac{q}{3.83424}\right)^{\frac{11}{2}} \ldots$$

(23)

Clearly, the series on the right of Eq. (23) is fast converging for small values of q. For large q, adopting $x = T^4$, $g(x) = x^{-1/4}$, $a = \pi/(128q)$, $\theta = \pi/(4q)$

and $\phi(x) = x^{1/4}/3 + x^{1/2}$ in Eq. (22) and applying Lagrange theorem with some simplifications yields

$$D_* = \frac{4}{3} + \left(\frac{q}{0.02454}\right)^{\frac{1}{4}} + \left(\frac{1.24112}{q}\right)^{\frac{1}{4}} + \left(\frac{0.55161}{q}\right)^{\frac{1}{2}} - \left(\frac{0.94098}{q}\right)^{\frac{3}{4}} + \left(\frac{0.37144}{q}\right)^{\frac{5}{4}} + \left(\frac{0.57154}{q}\right)^{\frac{3}{2}} - \left(\frac{0.81428}{q}\right)^{\frac{7}{4}} + \left(\frac{0.56270}{q}\right)^{\frac{9}{4}} + \left(\frac{0.74826}{q}\right)^{\frac{5}{2}} - \left(\frac{0.92999}{q}\right)^{\frac{11}{4}} + \left(\frac{0.72183}{q}\right)^{\frac{13}{4}} \ldots$$

(24)

It can be seen that the series on the right hand side is fast converging for large values of q. Considering nine terms in each equation, the absolute percentage errors occurring in Eqs. (23) and (24) are plotted in Figure 3. A perusal of Figure 3 shows that the maximum error involved is about 0.1% that occurs at about q = 1.05. Thus Eq. (23) is valid for $q \le 1.05$, whereas Eq. (24) is valid for $q \ge 1.05$. Thus the Eqs. (23) and (24) are

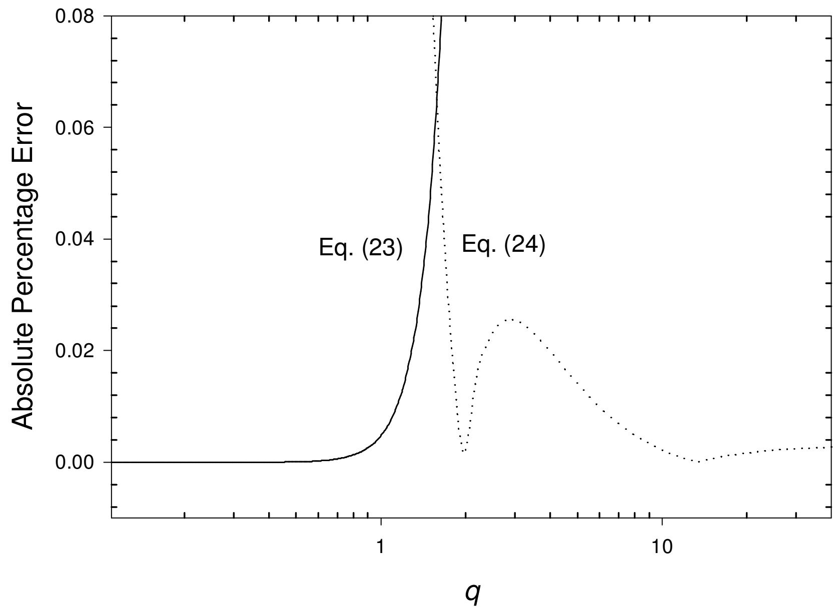

Figure 3. Absolute Percentage Error Curves for Eq. (23) and (24)

sufficiently accurate for all practical purposes.

Practical examples

The use of equations developed is illustrated by the following examples.

Example 1

As shown in Figure 4, drilling mud has to be pumped down into an oil well of L = 2,450 m deep. The mud is to be pumped at a rate of Q = 0.003 m^3/s to the bottom of the well and back to the surface through a pipe having an effective diameter D = 0.1 m. The pressure head at the bottom of the well is h_b = 2,636 m. Find the pumping head required. The drilling mud has the following properties of a Bingham plastic: yield stress τ_o = 10 N/m^2; limiting (plastic) viscosity μ_∞ = 0.035 Pa-s; and mass density ρ =1,200 kg/m^3.

Solution

The velocity of the fluid flow: $V = 4Q/\left(\pi D^2\right)$ = 0.382 m/s. The rheological properties are: $\nu_\infty = \mu_\infty/\rho$ = 0.035/1,200 = 2.9167x10^{-5} m^2/s and $s_o = \tau_o/\rho$ = 10/1,200 = 8.3333x10^{-3} m^2/s^2. Using Eqs. (5) and (6), Re = 1,310; and He = 97,959. Swamee and Aggarwal (2011) gave following equations for critical Reynolds number Re_c up to which the flow remains laminar:

$$\mathrm{Re}_c = 2{,}100\left(1+\frac{\mathrm{He}}{3{,}600}\right)^{0.35} \qquad 1 \le \mathrm{He} \le 10^8$$

(25)

$$\mathrm{Re}_c = 161\mathrm{He}^{0.334} \qquad 10^8 \le \mathrm{He} \le 10^{12}$$

(26)

Using Eq. (25) for He = 97,959, Re_c = 6,759. As Re < Re_c, the flow is laminar. Considering He/Re = 97,959/1,310 = 74.7779 < 30. Thus Eq. (15) is applicable. Thus, using Eq. (15), f = 0.5855. Therefore, the pumping head h_o that the pump has to develop is

$$h_o = h_b - h_{atm} + \frac{fLV^2}{2gD} - L = 2636 - 8.6 + \frac{0.5855\times2450\times0.382^2}{2\times9.8\times0.1} - 2450 = 0.284.2\ \mathrm{m}$$

Example 2

Find the commercial size of steel pipe that can transport a coal-water slurry at 0.0442 m^3/s over 30 m when a power equivalent to a static head of 2 m is applied. The mass density and the viscosity of the slurry are 2,000 kg/m^3 and 0.2 kg/m/s respectively; and the yield stress is 80 Pa.

Solution

In this case $S_f = h_f/L$ = 2/30 = 0.06667; $\nu_\infty = \mu_\infty/\rho$ = 0.2/2,000 = 10^{-4} m^2/s; and $s_o = \tau_o/\rho$ = 80/2,000 = 0.04

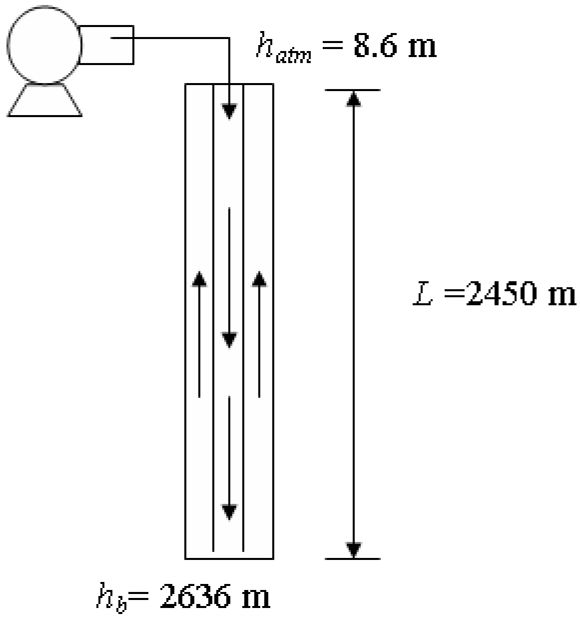

Figure 4. Example 1

m^2/s^2. Adopting g = 9.8 m/s^2 in Eq. (16), $q = 10^{-4}(9.8*0.06667)^3 0.0442/0.04^4 = 0.4815$, which is less than 1.05. Thus, Eq. (23) is applicable. Using Eq. (23) for nine terms D_* = 4.7006. Using Eq. (17) D = 4.7006x0.04/(9.8x0.06667) = 0.2878 m. Provide commercially available pipe of diameter 0.3 m. Using Eq. (5) Re = 4x0.0442)/(0.3πx10^{-4}) = 1,876. Eq. (6) gives He = 0.09x0.04/10^{-8} = 360,000. Now using Eq. (25) Re_c = 10,562. As Re < Re_c, the flow is laminar.

CONCLUSIONS

From the foregoing developments, the following conclusions are drawn for flow of Bingham plastic fluids in circular pipes:

1. The exact equations for the friction factor problem are given by Eqs. (12) and (15).
2. The exact equations for pipe diameter problem are given by Eqs. (23) and (24).
3. Use of the equations is illustrated by examples.

REFERENCES

Bralts VF, Kelly SF, Shayya WH, Segerlind LJ (1993). Finite element analysis of micro-irrigation hydraulics using a virtual emitter system. Trans. Am. Soc. Agric. Eng, 36: 717–725.

Danish M, Kumar S, Kumar S (2011). Approximate explicit analytical expressions of friction factor for flow of Bingham fluids in smooth pipes using Adomian decomposition method. Commun Nonlinear Sci. Numer. Simulat. 16: 239–251.

Darby R (1996). Chemical Engineering Fluid Mechanics., Marcel Dekker, New York City, New York, pp164-169.

Darby R, Melson J (1981). How to predict the friction factor for flow of Bingham plastics. Chemical Engineering, 28: 59-61.

Goudar CT, Sonnad JR (2007). Explicit Reformulation of the Colebrook-White Equation for Turbulent Flow Friction Factor Calculation. Industrial and Engineering Chemistry Research, 46: 2593-260.

Haaland SE (1983). Simple and Explicit Formulas for the Friction Factor in Turbulent Flow. J. Fluids Eng. (ASME) 103: 89–90.

Sablani SS, Shayya WH, Kacimov A (2003). Explicit calculation of the friction factor in pipeline for of Bingham plastic fluids: A neural network approach. Chem. Eng. Sci. 58: 99-106.

Serghides TK (1984). Estimate friction factor accurately. Chemical Engineering 91: 63–64.

Swamee PK, Aggarwal N (2011). Explicit equations for laminar flow of Bingham plastic fluids. J. Petrol. Sci. Eng. 76:178–184.

Swamee PK, Jain AK (1976). Explicit equations for pipe flow problems. J. Hydraul. Eng. ASCE, 102: 657-664.

Whittaker ET, Watson GN (1965). A course of modern analysis. Cambridge University Press, Cambridge, U.K.