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Full Length Research Paper

An examination of freshman student's mathematical proof skills

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Abstract

This article investigates the capacity of freshman university students to construct deductive proofs. Mathematical proving is a necessary acquisition for students to understand mathematics analytically. Students who have the ability to prove can better interpret mathematical expressions. Thus, they may see better mathematical concepts underlying relationships by performing meaningful learning. One of the major goals of mathematics education is to offer students the ability to think. In this sense, mathematical proofs may help the development of skills in abstract. This study aims to analyze the mathematical proof skills of freshman students by asking them six questions about propositions. To this end, the responses of 106 participants involving proofs were classified in six categories, for which quantitative results are presented.

Keywords: Mathematical proof, cognitive development, mathematical reasoning.

INTRODUCTION

Mathematics is a scientific discipline in which cause-effect relationships underlying in events are demonstrated in meaningful logical chains. As in all scientific disciplines, one needs to demonstrate the validity of a given expression in mathematics. Otherwise, validity of an expression cannot be justified. One of the greatest aims of mathematics curriculum for students at the stage of abstract thinking is to raise individuals with the ability to logically explain the underlying meaning of concepts through cause-effect relationship and to implement the principles of induction and deduction. Nevertheless, numerous studies investigating student's capacity to construct proofs revealed that they had difficulties about mathematical proving (Baker and Campbell, 2004; Recio and Godino, 2001; Dubinsky, Elterman and Gong, 1988; Moore, 1994; Weber, 2001; Harel and Sowder, 1998; Morali, Ugurel, Turnuklu and Yesildere, 2006).

Baker and Campbell (2004) attempted to identify the deficiencies usually faced in proving mathematical propositions for university students. As a result, they found that the students had incorrect understanding of concepts, did not know about the ways of prove statements, did not spend enough time to discuss the underlying meanings of concepts during proving, failed to use mathematical terminology correctly. In their study with a large group of university students, Recio and Godino (2001) found that the students had certain shortcomings about proving propositions and thus, did not know how to construct a proof. Selden and Selden (1995) asked 61 students to translate informal mathematical statements into the formal language of analysis at the beginning of proof lessons, observing that less than 10% of the students managed to do this successfully. Dubinsky, Eltermann, and Gong (1988) showed that it was a very difficult and complicated process to draw meaning from a logical statement. In a study conducted with undergraduate students, Moore (1994) found that the students failed to explain a given expression due to their inadequate concept. The author also revealed that the students did not know how to start a proof. During interviews conducted with 16 high school mathematics teachers, Knuth (2002) demonstrated that these teachers' proof concept was limited and they had concern about proofs. Harel and Sowder (1998) classified three dimensions of student proof schemes: 1. Externally based proof schemes; 2. Empirical proof schemes; and 3. Analytical proof schemes. Weber (2001) distinguished between two categories of student's difficulties about proving: The first is that they do not have full concept knowledge required to complete a mathematical proof, while the second is their lack of understanding about theorems or concepts and their lack of knowledge of the systematic methods used in proving.

Another study of Weber (2006) graded student's difficulties in mathematical proof in three categories. They were, respectively: inadequate conception knowledge about mathematical proof, misunderstanding of a theorem, and inadequacy in developing strategies for proof.

Balacheff examined mathematical proof at three levels; i.e., pragmatic proof, intellectual proof, and demonstration. The lowest level includes 'pragmatic proofs', which are representations with examples; medium-level proofs are 'intellectual proofs', which are proofs constructed on the basis of formulation; and the highest level involves 'demonstrations'; i.e., proofs that need to be organized by a theory or use knowledge accepted by a community (Balacheff, 1988; cited in Ozer and Arikan, 2003, p. 1).

In their study with 220 high school students, Ozer and Arikan (2002) observed that most of the students were at pragmatic proof level according to the abovementioned proof levels of Balacheff. Morali et al. (2006) developed a questionnaire for measuring "mathematical proving" construction of 337 pre-service teachers who are studying in their first and final years in the Faculty of Education. They conclude that the pre-service teacher's opinions about proof construction were not fully formed, and they were not aware of the significance of proof construction in mathematics and mathematics teaching. In addition, the authors underline the need for conducting more studies on mathematical proving, calling for attempts to investigate and reveal the necessity of mathematical proofs and their effects on the development of mathematical thinking.

This article aims to examine the capacity of freshman university students to construct deductive proofs. At the end of the study, the student's stages of validating propositions were classified. For this purpose, the five questions in Table 1 were used to examine freshman university student's proving skills.

METHOD

Participants

This study was carried out during the fall semesters of 2010-2011 and 2011-2012 with 106 freshman students who just started their university education in Faculty of Education and they had last took a mathematics course at high school. The student group participating in the study was randomly selected from among the departments recently starting university education. The students were not provided with any information about the proof concept prior to the study.

MATERIALS

Each student was asked to prove the validity of the five propositions given in Table 1 (third, fourth and fifth questions were prepared by Ozer and Arikan (2002) in order to determine the proof levels of high school students in their study).The student's proof skills will be classified by aid of analyzing five proposition's proof. Considering the opinions of three field experts, these questions were approved for use in the present study.

Data analysis

In this study, each student's response to 5 question s in table 1 had been investigated by researcher. Deficiencies were classified in 6 categories in the answers. Descriptive analysis was made of student's answers to these categories has been reached. Some of those categories found Recio and Godino (2001), Baker and Campbell (2004) is similar in types. The frequencies and percentages of the student's responses for each question are obtained according to the response six categories.

RESULTS

Classification of the responses

The responses to the questions in Table 1 of 106 students who had just started undergraduate studies at university and had gained knowledge of proofs and logic in high school were subjected to analysis, and six categories were revealed in line with the opinions of field experts. These categories classify the student's proving skills through their responses to prove the five propositions given in Table 1. The categories were not simply considered as the reasons behind the student's failure to construct proofs. In this perspective, the following categories except for the fifth and sixth could also be taken as the cause of student's incapacities to construct proofs, while the fifth and sixth categories may be regarded as cases of student's proving skills. Taken these categories as a whole, basic information could be obtained about the nature of the student's proving skills.

Table 1. A list of the questions used in the study.

1. Let a be a positive integer. If a^2 is even, then a is also even. Demonstrate it.
2. Demonstrate that $\sqrt{2}$ is an irrational number.
3. Let b and c be divisible by a. Then (b+c) is also divisible by a.
4. If $a \neq 0$, demonstrate that $\left(a^m\right)^n = a^{mn}$ for each $m, n \in N$.
5. Show that the sum of two odd numbers is an even number.

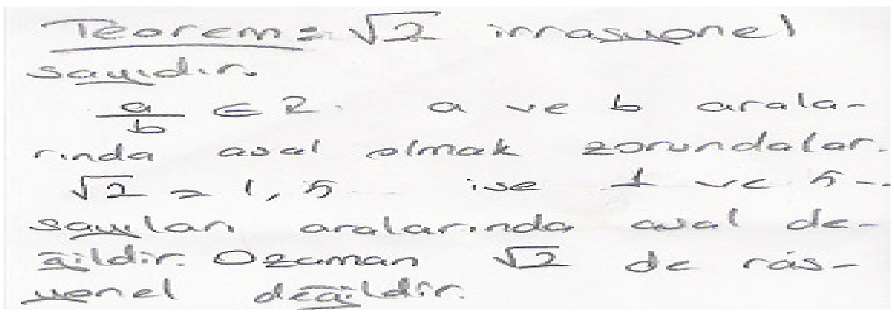


Figure 1. A student's response for category 1.

Lack of mathematical reasoning and misunderstanding mathematical definitions

Mathematical reasoning is a process by which conclusions are drawn from facts, judgments and propositions. From the student's responses in this category, it is clear that they did not know how to arrive at the result using the data given in the statement. In other words, the cause-effect chain did not work. In addition, incorrect mathematical definitions were also observed in the responses in this category. For instance, one such mistake was to define all decimal numbers as irrational numbers. Figure 1 shows a student's response in this category.

As shown in Figure 1, the student's response to the second question is as follows: "$\frac{a}{b} \in \Re$, a and b have to be relatively prime. If $\sqrt{2} = 1.5$ then 1 and 5 are not relatively prime. Then $\sqrt{2}$ is not rational."

The student's response in Figure 1 displays misunderstood mathematical definitions as well as serious mathematical errors. A brief interview was made with the student about the above response, which is given in the following dialogue.

Researcher (R): Why does $\sqrt{2}$ equal 1,5 ?
Student (S): If 1.5 is multiplied by 1.5, the result is 2.
R: 1.5x1.5 does not equal two.
S: Right, but I can say 2.
R: You say 1 and 5 are not relatively prime. Can you tell me what the condition is for being relatively prime?
S: I don't remember.

Lack of knowledge about axiomatic proof techniques and shortcomings in mathematical definitions and propositions

It was observed that the students could not use systematic proof methods (direct, indirect, contradiction, etc.) in their responses due to certain problems in induction and deduction steps. They do not know what the concepts of if … then ($\Rightarrow$) and if and if only ($\Leftrightarrow$) mean. They failed to display any discipline (rigor) in reaching the result mathematically using the data at hand. The following Figure 2 presents a response in this category.

As seen in Figure 2, the student's response to the first question in Table 1 is as follows. "If a^2 is even, then a is also even. Let a=2k. a^2=4k^2 If a number is multiplied by an even number, the result is even."

The student failed to produce the desired result due to the shortcomings in his proof techniques.

Discontinuity in one's perspective toward number system (proof construction using numerical examples)

This category involves proofs claiming that a proposition will always be true by validating it through a numerical

Figure 2. A student's response from category 2.

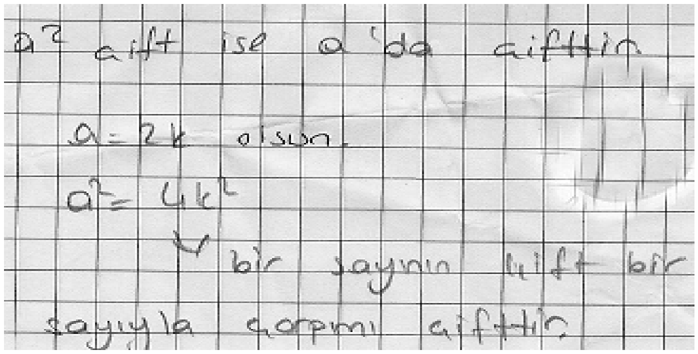
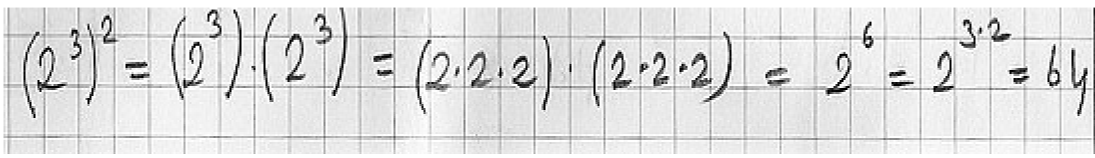


Figure 3. A student's response for category 3.



example. For instance, a student thinks only 2 or 4 as an even number but assuming $\forall n \in \mathrm{N}$ she/he cannot model all even numbers as 2n. In other words, she/he has a discontinuous perspective toward number system. One student in this category provided the response given in Figure 3 above.

Here, by validating the proposition by only a single numerical example, it was claimed that the proposition would always be true. The following interview was made with the student about this response.

R: If $a \neq 0$, you showed the validity of $(a^m)^n = a^{mn}$ for each m, $n \in$ N by using a numerical example. Can you tell that the proposition would be valid for all numbers simply by obtaining the result using a single numerical example?

S: Yes, I can. Because the result is the same for all numbers you write for a, m, n.

Proof construction without using mathematical symbolization (using verbal statements)

This category includes responses that verbally examine cause-effect relationship. The responses caught a relation between what is given and what is desired in the propositions; however, the mathematical model was formulated in verbal language, or there was no mathematical symbolization. A student's response in this category is as follows. "If we take a to be an odd number, an odd number multiplied by itself, which is odd, gives an odd number. Then, if the square of a number is even, the number must be even." The proof is verbally correct but it lacked mathematical representations. Moreover, the student has the skill to construct contrapositive (indirect) proof. The researcher wondered whether this student had acquired this particular proof skill during high school years or she/he just had it intuitionally. So she was asked why she came up with such a response. The student said: "If a^2 is even, so is a because if a is odd, its square has to be odd too. Then, if a number's square is even, the number itself is also even." When she was asked if any particular method was used in this response, she replied, "No, if you think reversely, you will see the result". Clearly, the student did not have any systematic knowledge of the indirect proof method in her responses, but she intuitionally reached de-Morgan's rule $((p \Rightarrow q)' \equiv (q' \Rightarrow p'))$. Among the 106 students, this student was the closest to the desired proof for the first question in Table 1 using the proof technique in question.

Attempts to construct proofs through correct mathematical definitions, propositions and theorems but did not provide a proper set-up by defining what domain the variable come from

The responses in this category used certain mathematical definitions and propositions. The students attempted to work out proofs using certain definitions or propositions they had learned before. Yet, as a result of some errors, they failed to obtain the desired proofs. Figure 4 shows a student's response for this category. Here, the student wrote that if b is divided by a with no remainder, b=ak and if c is divided by a with no remainder, then c=ax. The following interview was conducted with the student who gave this response.

R: Which set of numbers do k and x belong to?

S: They are natural numbers.

R: Couldn't they be integers?

S: Yes, they could.

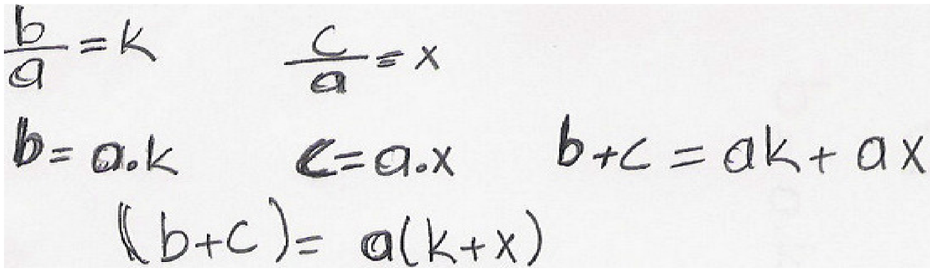


Figure 4. A student's response from category 5.

Table 2. Frequencies and percentages for the response categories.

Response	Q 1		Q 2		Q 3		Q 4		Q 5		Mean
Category	Freq.	%	Freq.	%	Freq.	%	Freq.	%	Freq.	%	%
1	56	52.8	64	60.3	8	4.5	4	3.7	-	-	24.2
2	24	22.6	24	22.6	4	3.7	-	-	6	5.6	10.9
3	12	11.3	-	-	38	35.8	72	67.9	66	62,2	36.2
4	2	1.8	-	-	6	5.6	-	-	4	3.7	2.2
5	4	3.7	6	5.6	46	43.3	20	18.8	24	22.6	18.8
6	-	-	-	-	2	1.8	10	9.4	4	3.7	2.9
No Response	8	4.5	12	11.3	2	1.8	-	-	2	1.8	3.8
Total	106	100	106	100	106	100	106	100	106	100	100

R. Then why didn't you write in your response that k and x are integers?
S: I did not feel the need to do so.
Although mathematical definitions and propositions used in this proof, x and k were not defined well in a domain.

A cause-effect chain that uses correct mathematical symbolization as well as other theorems and propositions

The students, who are in this case, reflected systematic proof techniques to their responses using the mathematical language. The researcher asked a student who is in this case, how you works mathematics course. The student said that he had interest in mathematics and enjoyed solving mathematics problem for this reason he took of advantage library books.

Quantitative results

Table 2 presents the frequencies and percentages of the student's responses for each question in Table 1 according to the response categories given in Section 3.1. Of the 106 students participating in the study, only 2.9% have the proving skill, 97.1% of the students had problems while constructing proofs, and 3.8% of them preferred not to provide any response. When asked why they did not come up with a response during the study, most of these students replied that they had been afraid to construct incorrect proofs.

36% of the students considered validating the propositions with numerical examples as a proof method. Among the responses categorized in Section 3.1, the highest rate of responses belonged to the third category (discontinuity in one's perspective toward number system). We believe that the most significant reason behind the student's assumption that their proof was complete when they numerically validated given propositions might be their inability to shift to the stages of analysis, synthesis and evaluation over the problems. One fundamental goal of mathematics education is to help students acquire abstract thinking skill. This skill might be expected to take shape along with the stages of analysis, synthesis and evaluation. Thus, the students in the response category 3 lack the abstract thinking skill. Consequently, their mathematical proofs were simply limited to the stage of validation by examples.

The third category as the most preferred response category was immediately followed by the first category with a rate of 24.2% of the student responses; that is, the lack of mathematical reasoning and serious mathematical errors. In this category of responses, the students were observed to have errors about mathematical thinking, performing operations, and basic definitions.

Although 10.9% of the students attempted to construct proofs by using the concepts of if … then ($\Rightarrow$) and if and if only ($\Leftrightarrow$) in their responses, they misused the mathematical meanings for these statements. For instance, by assuming the result required from a proposition containing $\Rightarrow$ to be true, they obtained the initially given data. This suggests that the students had difficulty in implementing proof methods.

In 2.2% of the responses obtained in the study, mathematical statements were correctly formulated using verbal language but no mathematical representations were provided. Students who produced such responses presented mathematical reasoning, the rings of the chain between cause and effect through logical thought.

18.8% of the student responses attempted to construct proofs using correct mathematical definitions, propositions and theorems; yet, did not provide a proper set-up by defining what domain the variable come from. Although definitions and mathematical operations were correctly given, the proofs were not satisfactory due to problems about understanding the number system involved (e.g., for x divided by a, a=kx was written but there was no mention of which set k belongs to).

CONCLUSIONS

The present study was carried out with 106 freshman students who just started their university education at the Faculty of Education. The students were asked to answer the five questions given in Table 1. Their responses were categorized as seen in Section 3.1. Six categories of student responses were identified as a result of this categorization.

Recio and Godino (2001) classified the proof frameworks of a group of students according their responses to questions involving geometry and arithmetic problems and came up with five categories of student responses. This study by Recio and Godino (2001) (University of Córdoba, Spain) was taken as a reference for the present study, though not strictly. Recio and Godino (2001) classified student responses over the responses to an arithmetic problem and a geometric one, dividing the student's proof frameworks into the following five categories:

1. Students with very deficient responses (confused, incoherent),
2. Students who checked the proposition with examples without serious mistakes,
3. Students who asserted the validity of the proposition with examples,
4. Students who justified the validity of the proposition by using well-known theorems or propositions, by means of partially correct procedures,
5. Students who gave substantially correct proofs using appropriate symbolization.

Similarities could be found with these results of Recio and Godino and those of the present study and they can be summarized as follows. Recio and Godino's (2001) results concerning their second and third categories – checking the proposition with examples without serious mistakes and asserting the validity of the proposition with examples – seem to be similar to the third category of student responses in the present study, which is "discontinuity in one's perspective toward number system (proof construction using numerical examples)". The responses of most (36%) students in this study fall in this category. One result in Baker and Campbell's (2004) research was that university students lack the knowledge of proof methods. In parallel, the second category of the present study, "lack of knowledge about axiomatic proof techniques and shortcomings in mathematical definitions and propositions", bears resemblance to Baker and Campbell's (2004) result. Selden and Selden (1995) highlighted the difficulties that students experience in using formal mathematical language while constructing mathematical proofs. In the present study, only a few (2.9%) of the student's responses were classified under the sixth category – "a cause-effect chain that uses correct mathematical symbolization as well as other theorems and propositions". With its results, this study revealed that most students had difficulty in constructing proofs and serious problems in their mathematical proof skills, a result that was also obtained by Ozer and Arikan (2002) and Almedia (2003) in their research.

Mathematical reasoning ability is initiated when the student starts thinking concretely. Concrete thinking skill is supposed to develop during the first elementary grade. Students need appropriate activities for mathematical reasoning to develop. They should be allowed to construct mathematical concepts on their own. This way, the relationship between causes and effects will be fully revealed. Children's concept of proving can develop during the pre-school period. For instance, children should not be simply asked to memorize numbers when they are introduced with the concept of numbers during pre-school period. The curriculum also requires their acquisition of the concept of plenitude denoted by numbers and later, conservation of numbers. So in pre-school years, the meaning of numbers is constructed by the child within a cause-effect relationship. Elementary-level children are asked to learn mathematical concepts within a cause-effect relationship. In these years, children are at the concrete operational stage, as Piaget identified. Mathematical proofs require materials. For example, after students are introduced with the meaning denoted by a proper fraction using whole-part relationship, demonstrations on a numerical axis would make much more sense. Addition and subtraction operations made with students who lack an understanding of proper fractions will only result in their memorizing the operations. Memorizing is the opposite of the concept of learning. Implementing mathematical methods without the knowledge of their causes will not help students acquire logical thinking and creativity. In brief, children's mathematical reasoning and proving skills will develop if the activities required for every development stage are carried out with the order given in the literature. Otherwise, memorizing individuals who do not investigate the causes will be trained.

If learning is to have access to new knowledge using previous knowledge, proving is to demonstrate the validity of a new proposition or the rationale of a theorem using a mathematical language and previous mathematical definitions, propositions and theorems. If new knowledge is constructed without any support from related sub-knowledge, then it is learned by rote. Rote knowledge is not only useless for student's cognitive development, but it can also slow down or intercept cognitive development. Mathematical proofs should be employed at every level of mathematics education. Otherwise, this skill will not be learned due to the behaviors acquired. Students should learn the reasons behind every characteristic, proposition, theorem, in short, every operation performed in mathematics. Mathematics courses should basically aim at training students who are curious to know how formulae were obtained and possess proving skills (students who know the rules of constructing proofs with sophisticated mathematical reasoning skills, who follow down the cause-effect chain, and can implement induction and deduction), instead of students who simply use a given formula to arrive at a result.

Increased number of studies in this field will be effective in revealing student's existing proof frameworks. With the help of programs updated in the light of such information, individuals will be trained in the future who have better mathematical proving and reasoning frameworks. Thus, there will be a greater number of logical-thinking, unbiased individuals.

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